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TWO MULTIOBJECTIVE PROBLEMS FOR STOCHASTIC DEGENERATE PARABOLIC EQUATIONS*

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Abstract. This paper is devoted to studying two multiobjective problems for stochastic degenerate parabolic equations. The first one is a hierarchical control problem, in which the controls are classified into a pair of leaders and a pair of followers. For each pair of leaders, a Nash equilibrium is searched for a noncooperative game problem. The aim of the pair of leaders is to achieve null controllability of the system. The other multiobjective problem is an inverse initial problem under a Nash equilibrium strategy. In contrast to the classical inverse initial problem, an optimization problem for stochastic degenerate parabolic equations is first investigated. Then, the conditional stability of determining initial information is derived through terminal observation.

Key words. stochastic degenerate parabolic equation, null controllability, optimization problem, conditional stability, Carleman estimate

MSC codes. 93B05, 93B07

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1. Introduction.

In most classical control problems, a system usually has only one objective or task. For example, the aim is to find a control to attain a given target or to determine unknown information of the system by local observation. However, in practice, an interesting situation will arise when several different (even contradictory) objectives are studied simultaneously, such as economic, transportation, and engineering systems [9, 26, 33]. A variety of control strategies appear, based on the characteristics of multiobjective control problems in economics and game theory [25, 27, 29]. A related question is whether one can direct the system to a desired state by exerting controls corresponding to the strategies.

In the multiobjective case, the system may involve multiple active actors, whose behaviors are motivated by self-interest. The individual rationality may result in strategic behaviors to pursue their own interests. Sometimes, individual behavior has a leader-follower relationship, and the corresponding strategy was introduced in the work of Stackelberg [29] to describe this situation. The general idea of this strategy is a hierarchical game where players compete with each other. The leaders take action first, and the followers make the corresponding response to the action of the leaders. There may be many followers, each of them with a specific objective. And, the followers tend to achieve a Nash equilibrium. A Nash equilibrium is a combination

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of strategies in which no player can benefit from changing strategies alone. In other words, when the others do not change their strategies, no one can get more benefit by only changing his own strategy, and then the combination of strategies is a Nash equilibrium.

As a typical class of diffusion equations, degenerate parabolic equations can describe many different physical phenomena, such as Budyko–Sellers equations that model the interaction between large ice masses and solar radiation [6]. Stimulated by the need to take into account random effects in practical problems, stochastic processes naturally replace deterministic functions in a mathematical way, and the associated model becomes a stochastic degenerate parabolic equation. This paper is devoted to investigating the controllability and inverse problem for a class of stochastic degenerate parabolic equations under a Nash equilibrium strategy.

Let $T > 0$, and let G_0, G_1, and G_2 be nonempty open subsets of $(0,1)$. Set $Q = (0,1)\times(0,T)$, and set $a(x) = x^\alpha$ for $\alpha \in [0,2)$ and $x \in [0,1]$. Denote by χ_{G_0} the characteristic function of the set G_0. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, on which a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined, so that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all $\mathbb{P}$-null sets in $\mathcal{F}$. Let $\mathcal{H}$ be a Banach space and $C([0,T];\mathcal{H})$ be the Banach space of all $\mathcal{H}$-valued strongly continuous abstract functions defined on $[0,T]$. We denote by $L^2_{\mathbb{F}}(0,T;\mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{H})}) < \infty$ with the canonical norm, $L^\infty_{\mathbb{F}}(0,T;\mathcal{H})$ denotes the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted essentially bounded processes, and $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathcal{H}))$ denotes the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];\mathcal{H})}) < \infty$.

Consider the following one-dimensional stochastic degenerate parabolic equation:

$$
\begin{cases}
\mathrm{d}y - \left(x^\alpha y_x\right)_x \mathrm{d}t = (a_1 y + u_1\chi_{G_0} + v_1\chi_{G_1} + v_2\chi_{G_2})\mathrm{d}t & \\
\qquad\qquad\qquad + (a_2 y + u_2)\mathrm{d}W(t) & \text{in } Q,\\
\begin{cases} y(0,t) = 0 & \text{if } 0 \leq \alpha < 1,\\ \left(x^\alpha y_x\right)(0,t) = 0 & \text{if } 1 \leq \alpha < 2\end{cases} & \text{on } (0,T),\\
y(1,t) = 0 & \text{on } (0,T),\\
y(x,0) = y_0(x) & \text{in } (0,1),
\end{cases}
\tag{1.1}
$$

where y_x denotes the weak derivative with respect to the spatial variable x, (u_1,u_2) is the pair of leaders, (v_1,v_2) is the pair of followers, $y_0 = y_0(x)$ denotes the initial data, and $y = y(\cdot,\cdot;u_1,u_2,v_1,v_2)$ is the state variable.

First, we study a controllability problem in the sense of the Stackelberg–Nash strategy for (1.1). In fact, this type of controllability means that for each pair of leaders, a Nash equilibrium pair needs to be found, and the goal of the pair of leaders is to achieve controllability. The decisions of the followers are influenced by the decisions of the leaders.

The controllability problems in the sense of the Stackelberg–Nash strategy come from practice. For example, in order to remedy market failure and promote cooperation, the government can regulate macroeconomics through legislation and policies. Using the tools and methods of game theory, we can model the macroeconomic system as a noncooperative dynamic game involving the government, multiple enterprises, and the market. The government can influence the dynamics of the system by formulating certain policies. Enterprises and the market groups can pursue their own interests based on these policies. If we take policy as a regulator of a higher level, then it can reach some satisfactory macroeconomic states by adjusting the Nash equilibrium formed by enterprises and the market.

When the model is described by a distributed parameter system, we refer to [1, 2, 3, 8, 11, 15, 16, 28] for some known works on Stackelberg–Nash type controllability problems. In the stochastic case, there have been rich works on the leader-follower hierarchical control problems on Stackelberg games ([24, 31, 34, 37] and references therein).

In [37], the authors focused on stochastic differential equations. They formulated controllability problem of stochastic game-based control systems in the general stochastic nonlinear framework and obtained some explicit necessary and sufficient algebraic conditions on the controllability of the Nash equilibrium for linear stochastic systems. In this paper, we focus on stochastic degenerate parabolic equations. We will first present the existence and uniqueness of the Nash equilibrium for any given pair of leaders (see Theorem 3.1), and then prove the null controllability in the sense of the Stackelberg–Nash strategy for (1.1) (see Theorem 2.1), by constructing an appropriate observability estimate (see Theorem 3.2).

The other multiobjective problem is an inverse problem for the stochastic degenerate parabolic system (1.1) under a Nash equilibrium strategy. This type of inverse problem has many practical applications. For example, if y represents the concentration of chemicals, we want to keep the concentration and the rate of change in concentration close to their desired values by adding water to or evaporating it from the chemical, and at the same time, the initial concentration is expected to be determined by the observation at the terminal time.

There have been numerous results on inverse problems for deterministic and stochastic partial differential equations ([13, 20, 21, 32, 36] and the references therein). Different from the classical inverse problems, the inverse problem studied in this paper is a combination of the classical one and game theory. The goal of controls is on the Nash equilibrium. Meanwhile, we want to determine the initial value from the known observation information. For this purpose, an interpolation inequality is proved to determine initial value through the observation at the terminal time (see Theorem 2.6).

In order to deal with the inverse problem, we establish quantitative uniqueness estimates for the coupled stochastic forward-backward equations. However, the classical inverse problems only study the forward equations. In this paper, the key to solving the inverse problem is the Carleman estimates for stochastic degenerate parabolic equations. By choosing a space-independent weight function, the desired Carleman estimates can be established by a weighted identity method (see Theorems 4.1 and 4.2) and a duality technique (see Theorem 4.3), respectively.

The main contributions of this paper are as follows:

• Compared with [37], we study the Stackelberg–Nash type controllability problem for the stochastic degenerate parabolic equation by the classical duality argument. The key is to transform the origin hierarchical control problem into the observability estimate of the coupled forward-backward stochastic partial differential equations. Compared with existing results, the followers in this paper achieve more goals, and the observation domains of the followers are less restricted. The energy functionals for followers can be more general and contain the gradient of the state. Since drift terms in the coupled equations belong to a Sobolev space of negative order, the problem is much more complicated.

• The inverse problem for a stochastic degenerate parabolic equation under a Nash equilibrium strategy is transformed into an inverse problem for the coupled forward-backward stochastic partial differential equations. Due to the appearance of coupling terms and the limitations of observation information, we establish new

Carleman estimates to overcome the difficulties. And, a quantitative estimate of the initial value is obtained by the Carleman estimates.

The rest of this paper is organized as follows. Section 2 presents the main concepts and results of this paper. Section 3 is devoted to characterizing the Nash equilibrium and proving the Stackelberg–Nash type controllability result. The result of conditional stability under a Nash equilibrium strategy is proved in section 4. Finally, section 5 summarizes the whole paper.

2. Problem formulation. In this section, we give some preliminary and the main results. First, some notations are introduced. For $0 \le \alpha < 1$, define the Hilbert space $H^1_\alpha(0,1)$ as follows:

$$H^1_\alpha(0,1) = \Big\{ \, y \in L^2(0,1) \;\Big|\; y \text{ is absolutely continuous in } [0,1], \\ x^{\frac{\alpha}{2}} y_x \in L^2(0,1) \text{ and } y(0) = y(1) = 0 \, \Big\}.$$

For $1 \le \alpha < 2$, $H^1_\alpha(0,1)$ is defined as follows:

$$H^1_\alpha(0,1) = \Big\{ \, y \in L^2(0,1) \;\Big|\; y \text{ is locally absolutely continuous in } (0,1], \\ x^{\frac{\alpha}{2}} y_x \in L^2(0,1) \text{ and } y(1) = 0 \, \Big\}.$$

By [6, 14], for any initial data $y_0 \in L^2(0,1)$, $a_1, a_2 \in L^\infty_{\mathbb{F}}(0,T;L^\infty(0,1))$, $(u_1,u_2) \in U_1 \triangleq L^2_{\mathbb{F}}(0,T;L^2(G_0)) \times L^2_{\mathbb{F}}(0,T;L^2(0,1))$, and $(v_1,v_2) \in U_2 \triangleq L^2_{\mathbb{F}}(0,T;L^2(G_1)) \times L^2_{\mathbb{F}}(0,T;L^2(G_2))$, (1.1) admits a unique solution $y \in \mathcal{H} \triangleq L^2_{\mathbb{F}}(\Omega;C([0,T];L^2(0,1))) \cap L^2_{\mathbb{F}}(0,T;H^1_\alpha(0,1))$.

Let us describe the controllability problem in the sense of the Stackelberg–Nash strategy. For $i = 1,2$, let $\mathcal{O}_i, \widetilde{\mathcal{O}}_i \subseteq (0,1)$ be the nonempty open sets, which represent the observation domains. For any given pair of leaders $(u_1,u_2) \in U_1$, define the (secondary) cost functionals

$$J_i(v_1,v_2) = \frac{\alpha_i}{2}\left(\mathbb{E}\int_0^T\int_{\mathcal{O}_i}|y - y_i^*|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{\widetilde{\mathcal{O}}_i}|x^\alpha y_x - y_{i+2}^*|^2\mathrm{d}x\mathrm{d}t\right) \\ + \frac{\mu_i}{2}\mathbb{E}\int_0^T\int_{G_i} v_i^2\mathrm{d}x\mathrm{d}t, \quad i = 1,2,$$

where α_i and μ_i are positive constants, $(y_i^*, y_{i+2}^*) \in (L^2_{\mathbb{F}}(0,T;L^2(0,1)))^2$ are given functions, and $y = y(\cdot,\cdot;u_1,u_2,v_1,v_2)$ is the solution of (1.1) corresponding to the pair of leaders (u_1,u_2) and the pair of followers (v_1,v_2).

For any given pair of leaders, we need to find a Nash equilibrium with a noncooperative optimization problem. In other words, the main objective of the pair of followers is to derive the state y and its derivative $x^\alpha y_x$ close to the desired states (y_i^*, y_{i+2}^*) in the corresponding observation domains, with the cost as little as possible for the pair of followers.

Indeed, a pair of followers $(\overline{v}_1,\overline{v}_2) \in U_2$ is called a Nash equilibrium pair if

$$\begin{aligned} &J_1(\overline{v}_1,\overline{v}_2) \le J_1(v_1,\overline{v}_2) \quad \forall\, v_1 \in L^2_{\mathbb{F}}(0,T;L^2(G_1)) \\ &\text{and } J_2(\overline{v}_1,\overline{v}_2) \le J_2(\overline{v}_1,v_2) \quad \forall\, v_2 \in L^2_{\mathbb{F}}(0,T;L^2(G_2)). \end{aligned} \tag{2.1}$$

This means a pair of followers is the Nash equilibrium pair if and only if the strategy chosen by any player is optimal, when all other players' strategies are determined. Then, define the main cost functional

$$J(u_1,u_2)=\frac{1}{2}\mathbb{E}\int_Q (u_1^2\chi_{G_0}+u_2^2)\mathrm{d}x\mathrm{d}t \quad \forall\,(u_1,u_2)\in U_1.$$

Let $\mathfrak{F}$ represent the set of the pair of leaders $(\overline{u}_1,\overline{u}_2)\in U_1$ satisfying the following condition:

$$y(\cdot,T;\overline{u}_1,\overline{u}_2,\overline{v}_1,\overline{v}_2)=0 \text{ in } (0,1),\ \mathbb{P}\text{-}a.s. \tag{2.2}$$

The goal of the pair of leaders is to derive the solution of (1.1) to reach zero at time T with the minimum cost. It means that there exists a pair of leaders $(\overline{u}_1,\overline{u}_2)$ satisfying that

$$J(\overline{u}_1,\overline{u}_2)=\min_{(u_1,u_2)\in\mathfrak{F}} J(u_1,u_2). \tag{2.3}$$

We introduce the following assumption conditions on the coefficient a_2 in this paper:
(H_1) $a_2\in L^\infty_{\mathbb{F}}(0,T;W^{1,\infty}(0,1))$,
(H_2) $a_2=x^{\hat{\alpha}}a_3$ with $\hat{\alpha}\geq\frac{2-\alpha}{2}$ and $a_3\in L^\infty_{\mathbb{F}}(0,T;L^\infty(0,1))$.

In what follows, C denotes a positive constant which may be different from place to place. The first main result is the Stackelberg–Nash controllability of (1.1).

THEOREM 2.1. *For $i=1,2$, assume that $\mathcal{O}_i\cap G_0\neq\emptyset$, $(\mathcal{O}_1\cap G_0)\nsubseteq\widetilde{\mathcal{O}}_i$, $(\mathcal{O}_2\cap G_0)\nsubseteq\widetilde{\mathcal{O}}_i$, $\mu_1,\mu_2>0$ are sufficiently large, and the condition* (H_1) *or* (H_2) *holds. Also, suppose that* Case 1 : $\mathcal{O}_1=\mathcal{O}_2$, $\widetilde{\mathcal{O}}_1=\widetilde{\mathcal{O}}_2$, *or* Case 2 : $\mathcal{O}_1\neq\mathcal{O}_2$ *holds, and let $(y_i^*,y_{i+2}^*)\in(L^2_{\mathbb{F}}(0,T;L^2(0,1)))^2$ be equal to zero near T. Then, for any $y_0\in L^2(0,1)$, there exist a pair of leaders $(\overline{u}_1,\overline{u}_2)\in U_1$, and the associated Nash equilibrium $(\overline{v}_1,\overline{v}_2)\in U_2$, such that* (2.2) *and* (2.3) *hold.*

By the classical duality argument [23], the null controllability in Theorem 2.1 can be deduced to an appropriate observability estimate (see Theorem 3.2 below). From the observability estimate, we can get the above controllability result immediately. So, we omit the proof of Theorem 2.1 here.

Remark 2.2. The conditions for (y_i^*,y_{i+2}^*) $(i=1,2)$ can be relaxed to

$$\mathbb{E}\int_Q\widetilde{\rho}^2(t)\left(|y_i|^2\chi_{\mathcal{O}_i}+x^\alpha|y_{i+2}^*|^2\chi_{\widetilde{\mathcal{O}}_i}\right)\mathrm{d}x\mathrm{d}t<+\infty,$$

where

$$\widetilde{\rho}(t)=e^{C\ell(t)} \quad\text{and}\quad \ell(t)=\begin{cases}\frac{4^4}{T^8} & \text{for}\quad 0\leq t\leq\frac{T}{2},\\ \frac{1}{t^4(T-t)^4} & \text{for}\quad \frac{T}{2}\leq t\leq T.\end{cases} \tag{2.4}$$

Remark 2.3. In this theorem, we get the same result in two cases. In Case 2, we don't discuss the difference between $\widetilde{\mathcal{O}}_1$ and $\widetilde{\mathcal{O}}_2$. This is because conditions $(\mathcal{O}_1\cap G_0)\nsubseteq\widetilde{\mathcal{O}}_i$, $(\mathcal{O}_2\cap G_0)\nsubseteq\widetilde{\mathcal{O}}_i$ $(i=1,2)$ are true in the theorem.

Remark 2.4. In practice, there may be many followers in the system, but the ideas of Stackelberg–Nash type controllability are the same, and the results can be obtained similarly. For simplicity, we only consider two follower controls in this paper, and the number of leader controls cannot be reduced. It is very interesting to study the controllability of a stochastic partial differential equation with only one leader control. Until now, a positive result has been available only for some special cases of stochastic parabolic equation [19].

Remark 2.5. Two follower controls in this paper are active on the drift term. The same result remains true when they are imposed on the diffusion term, or both the drift term and the diffusion term.

Next, an inverse problem under a Nash equilibrium strategy is studied. This type of inverse problem is a combination of the classical one and game theory. We consider (1.1) without leader controls, and the goal of (v_1, v_2) is to achieve a Nash equilibrium, i.e., (2.1) holds. At the same time, we want to determine the initial value from the observation information.

The second result is the following interpolation inequality for the weak solution of (1.1) when the pair of leaders $(u_1, u_2) = (0, 0)$.

THEOREM 2.6. *Assume that* $(u_1, u_2) = (0, 0)$ *in* (1.1), $\mu_1, \mu_2 > 0$ *are sufficiently large,* $M_0 > 0$, *and the condition* (H_1) *holds. For any* $y_0 \in H^1_\alpha(0,1)$, $t_0 \in (0,T]$, *there exist positive constants* $\kappa \in (0,1)$ *and* C, *only depending on* α, t_0, *and* T, *and a Nash equilibrium* $(\overline{v}_1, \overline{v}_2) \in U_2$ *such that the associated weak solution* y *to* (1.1) *satisfies*

$$\mathbb{E}\int_0^1 y^2(x,t_0)dx \le C\left(\mathbb{E}\int_Q y^2 dxdt + M_0\right)^{\kappa}\left(\mathbb{E}\int_0^1 y^2(x,T)dx + M\right)^{1-\kappa}, \tag{2.5}$$

where $M = \mathbb{E}\int_Q (y_1^{*2}\chi_{\mathcal{O}_1} + y_2^{*2}\chi_{\mathcal{O}_2} + x^{2\alpha}y_3^{*2}\chi_{\widetilde{\mathcal{O}}_1} + x^{2\alpha}y_4^{*2}\chi_{\widetilde{\mathcal{O}}_2})dxdt$.

The key for obtaining the above result is the Carleman estimate, and the proof of the theorem is given in section 4. As a direct consequence of the above theorem, one has the following backward uniqueness result for (1.1).

COROLLARY 2.7. *Under the assumption of Theorem* 2.6, *if* $y(T) = 0$ *in* $(0,1)$, $\mathbb{P}$*-a.s., and the desired states* $y_i^* = 0$ $(i = 1,2,3,4)$ *in* Q, $\mathbb{P}$*-a.s., then there exists a Nash equilibrium* $(\overline{v}_1, \overline{v}_2) \in U_2$ *such that the associated solution* y *to* (1.1) *satisfies* $y(t) = 0$ *in* $(0,1)$, $\mathbb{P}$*-a.s. for all* $t \in [0,T]$.

As another direct consequence of Theorem 2.6, we get the following conditional stability for (1.1).

COROLLARY 2.8. *For any* $L > 0$, *set* $H_L = \{y_0 \in H^1_\alpha(0,1) \,|\, |y_0|_{H^1_\alpha(0,1)} \le L\}$. *Under the assumption of Theorem* 2.6, *for any* $y_0 \in H_L$, *there exists a Nash equilibrium* $(\overline{v}_1, \overline{v}_2) \in U_2$ *such that the corresponding solution to* (1.1) *satisfies*

$$\mathbb{E}\int_0^1 y^2(x,t_0)\mathrm{d}x \le C(L, M_0)\left(\mathbb{E}\int_0^1 y^2(x,T)\mathrm{d}x + M\right)^{1-\kappa},$$

where $M = \mathbb{E}\int_Q (y_1^{*2}\chi_{\mathcal{O}_1} + y_2^{*2}\chi_{\mathcal{O}_2} + x^{2\alpha}y_3^{*2}\chi_{\widetilde{\mathcal{O}}_1} + x^{2\alpha}y_4^{*2}\chi_{\widetilde{\mathcal{O}}_2})dxdt$. *That is, under the condition that the initial data* y_0 *is in a bounded set, there exists a Nash equilibrium, such that any state* $y(x,t)$ $((x,t) \in Q)$ *can be determined quantitatively by the terminal data* $y(T)$ *and the given states* y_i^* $(i = 1,2,3,4)$.

Remark 2.9. Different from Theorem 2.1, we do not need to discuss the different case of $\mathcal{O}_i$ or $\widetilde{\mathcal{O}}_i$ in Theorem 2.6. This is due to the fact that the given states y_i^* $(i = 1,2,3,4)$ are observation information, and local information of the solution is not required in the observation information.

3. Stackelberg–Nash controllability.

3.1. Nash equilibrium. In this subsection, we present the existence and uniqueness of the Nash equilibrium for any given pair of leaders and give a characterization of it. For any given $(u_1,u_2)\in U_1$, the pair $(\overline{v}_1,\overline{v}_2)\in U_2$ is the Nash equilibrium with respect to J_1 and J_2 if and only if

$$
\begin{aligned}
&J_1'(\overline{v}_1,\overline{v}_2)(v_1,0)=0 \quad \forall\, v_1\in L^2_{\mathbb{F}}(0,T;L^2(G_1)),\\
&\text{and}\ \ J_2'(\overline{v}_1,\overline{v}_2)(0,v_2)=0 \quad \forall\, v_2\in L^2_{\mathbb{F}}(0,T;L^2(G_2)),
\end{aligned}
\tag{3.1}
$$

where J_1', J_2' denote the Fréchet operators of J_1, J_2, respectively. From the above equivalence, the result of the Nash equilibrium is as follows.

THEOREM 3.1. *Assume that $\mu_1,\mu_2>0$ are sufficiently large, such that for any given pair of leaders $(u_1,u_2)\in U_1$, there exists a unique pair of Nash equilibrium $(\overline{v}_1,\overline{v}_2)\in U_2$ satisfying* (3.1) *and*

$$
\overline{v}_i=-\frac{1}{\mu_i}p_i\chi_{G_i},\ i=1,2,\ \mathbb{P}\text{-}a.s.,
\tag{3.2}
$$

where (y,p_i,P_i) is the solution of the following coupled stochastic forward-backward degenerate parabolic equations:

$$
\begin{cases}
\mathrm{d}y-\left(x^\alpha y_x\right)_x\mathrm{d}t=\left(a_1y+u_1\chi_{G_0}-\frac{1}{\mu_1}p_1\chi_{G_1}-\frac{1}{\mu_2}p_2\chi_{G_2}\right)\mathrm{d}t & \\
\qquad\qquad\qquad +(a_2y+u_2)\mathrm{d}W(t) & \text{in } Q,\\
-\mathrm{d}p_i-\left(x^\alpha p_{i,x}\right)_x\mathrm{d}t=\big[a_1p_i+a_2P_i+\alpha_i(y-y_i^*)\chi_{\mathcal{O}_i} & \\
\qquad\qquad\qquad -\alpha_i(x^{2\alpha}y_x-x^\alpha y_{i+2}^*)_x\chi_{\widetilde{\mathcal{O}}_i}\big]\mathrm{d}t+P_i\mathrm{d}W(t) & \text{in } Q,\\
\begin{cases} y(0,t)=p_i(0,t)=0 & \text{if } 0\le\alpha<1,\\ \left(x^\alpha y_x\right)(0,t)=\left(x^\alpha p_{i,x}\right)(0,t)=0 & \text{if } 1\le\alpha<2\end{cases} & \text{on } (0,T),\\
y(1,t)=p_i(1,t)=0 & \text{on } (0,T),\\
y(x,0)=y_0(x),\quad p_i(x,T)=0 & \text{in } (0,1).
\end{cases}
\tag{3.3}
$$

Proof. See Appendix A for the proof.

3.2. Controllability. From Theorem 3.1, for any pair of leaders $(u_1,u_2)\in U_1$, there exists a unique pair of followers is the Nash equilibrium pair. The solution of (1.1) corresponding to the above pairs of leaders and followers satisfies (3.3). Thus, the controllability problem in Theorem 2.1 can be transformed into finding controls $(\overline{u}_1,\overline{u}_2)\in U_1$ such that the corresponding solution (y,p_i,P_i) to (3.3) satisfies

$$
y(T;\overline{u}_1,\overline{u}_2)=0\ \text{in}\ (0,1),\ \mathbb{P}\text{-}a.s.,
$$

with minimum cost. In other words, $(\overline{u}_1,\overline{u}_2)\in\mathfrak{F}$ minimizes the following cost functional J, that is,

$$
J(\overline{u}_1,\overline{u}_2)=\min_{(u_1,u_2)\in\mathfrak{F}}J(u_1,u_2),
$$

where

$$
\begin{aligned}
\mathfrak{F}=\{(u_1,u_2)\in U_1\,\big|\,&y(\cdot,T;u_1,u_2)=0\ \text{in}\ (0,1),\mathbb{P}\text{-}a.s.,\\
&y(\cdot,\cdot;u_1,u_2)\ \text{is the solution to}\ (3.3)\}.
\end{aligned}
$$

In order to accomplish the above goal, we establish an observability inequality for the following coupled forward-backward equations ($i = 1, 2$), which are the adjoint equations of (3.3):

(3.4)
$$\begin{cases} -\mathrm{d}z - \big(x^\alpha z_x\big)_x \mathrm{d}t = \big[a_1 z + a_2 Z + \alpha_1 q_1 \chi_{\mathcal{O}_1} + \alpha_2 q_2 \chi_{\mathcal{O}_2} \\ \qquad - \alpha_1 \big(x^{2\alpha} q_{1,x}\big)_x \chi_{\widetilde{\mathcal{O}}_1} - \alpha_2 \big(x^{2\alpha} q_{2,x}\big)_x \chi_{\widetilde{\mathcal{O}}_2}\big]\mathrm{d}t + Z\mathrm{d}W(t) & \text{in } Q, \\ \mathrm{d}q_i - \big(x^\alpha q_{i,x}\big)_x \mathrm{d}t = \big(a_1 q_i - \frac{1}{\mu_i} z \chi_{G_i}\big)\mathrm{d}t + a_2 q_i \mathrm{d}W(t) & \text{in } Q, \\ \begin{cases} z(0,t) = q_i(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ \big(x^\alpha z_x\big)(0,t) = \big(x^\alpha q_{i,x}\big)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{cases} & \text{on } (0,T), \\ z(1,t) = q_i(1,t) = 0 & \text{on } (0,T), \\ z(x,T) = z_T(x), \quad q_i(x,0) = 0 & \text{in } (0,1). \end{cases}$$

Similar to the discussion in [23], by constructing appropriate variational problems, it can be proved that the required pair of leaders exists if and only if the solution to (3.4) satisfies the following observability estimate.

THEOREM 3.2. *For $i = 1, 2$, under the condition of Theorem 2.1, there exist a positive constant C, only depending on G_0, G_i, $\mathcal{O}_i$, $\widetilde{\mathcal{O}}_i$, T, α, α_i, and μ_i, and a weight function $\widetilde{\rho}^2 = \widetilde{\rho}^2(t)$ blowing up at $t = T$ such that for any $z_T \in L^2_{\mathbb{F}}(\Omega, \mathcal{F}_T, P; L^2(0,1))$, the following inequality holds true for the solution (z, Z, q_i) to (3.4):*

$$\int_0^1 z^2(0)dx + \sum_{i=1}^{2} \mathbb{E}\int_Q \widetilde{\rho}^{-2}\big(q_i^2 + x^\alpha q_{i,x}^2\big)dxdt \le C\mathbb{E}\int_Q \big(z^2\chi_{G_0} + Z^2\big)dxdt. \tag{3.5}$$

In order to prove Theorem 3.2, we need to establish some estimates for (3.4) in Case 1 and Case 2, respectively. First, the known Carleman estimates [18] for stochastic degenerate parabolic equations are given.

Consider the following one-dimensional forward stochastic degenerate parabolic equation:

$$\begin{cases} \mathrm{d}y - \big(x^\alpha y_x\big)_x \mathrm{d}t = f\mathrm{d}t + g\mathrm{d}W(t) & \text{in } Q, \\ \begin{cases} y(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ \big(x^\alpha y_x\big)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{cases} & \text{on } (0,T), \\ y(1,t) = 0 & \text{on } (0,T), \\ y(x,0) = y_0(x) & \text{in } (0,1), \end{cases} \tag{3.6}$$

where $y_0 \in L^2(0,1)$, $f \in L^2_{\mathbb{F}}(0,T; L^2(0,1))$, and $g \in L^2_{\mathbb{F}}(0,T; H^1_\alpha(0,1))$.

Let $d(\cdot) \in C^4([0,1])$ be a function satisfying $d(x) > 0$ in $(0,1)$, $d(0) = d(1) = 0$, and $d_x(x) \neq 0$ in $[0,1]\backslash\omega$, where ω is a nonempty subset of $(0,1)$. And, for $\lambda, s > 0$, introduce the weight functions

$$\begin{aligned} &\theta(x,t) = e^{-\lambda\sigma(x,t)}, \quad \sigma(x,t) = \gamma(t)\phi(x), \quad \gamma(t) = \frac{1}{t^4(T-t)^4}, \\ &\phi(x) = \xi(x)\frac{2 - x^{2-\alpha}}{(2-\alpha)^2} + [1 - \xi(x)]\left(e^{2s|d|_{C([0,1])}} - e^{sd(x)}\right), \end{aligned} \tag{3.7}$$

where $\xi \in C^\infty([0,1])$ satisfying that $0 \le \xi \le 1$ in $[0,1]$, $\xi(x) = 1$ in $[0, x^1]$, and $\xi(x) = 0$ in $[x^2, 1]$, $0 < x^1 < x^2 < 1$. Then, from the known Carleman estimate result (see Corollary 2.1 in [18]) for the forward equation (3.6), we have the following lemma.

LEMMA 3.3 (see [18]). *For any positive integer k, there exist two positive constants s_k and λ_k, depending only on k, α, G_0, and T, such that for the $s \geq s_k$ and $\lambda \geq \lambda_k$, the solution y to* (3.6) *satisfies*

$$\begin{aligned}&\mathbb{E}\int_Q \theta^2 \left(\lambda^{3+k}\gamma^{3+k}x^{2-\alpha}y^2 + \lambda^{1+k}\gamma^{1+k}x^\alpha y_x^2\right) dxdt\\&\quad\leq C(s)\mathbb{E}\int_Q \lambda^k\gamma^k\theta^2\left(\lambda^3\gamma^3y^2\chi_{G_0} + f^2 + \lambda^2\gamma^2x^{2-\alpha}g^2 + \lambda\gamma^{\frac{5}{4}}g^2 + x^\alpha g_x^2\right) dxdt\end{aligned}$$

for any $y_0 \in L^2(0,1)$.

Consider the following backward stochastic degenerate parabolic equation:

$$\begin{cases} \mathrm{d}h + \left(x^\alpha h_x\right)_x \mathrm{d}t = (f_1 + f_{2,x})\mathrm{d}t + H\mathrm{d}W(t) & \text{in } Q,\\ \begin{cases} h(0,t) = 0 & \text{if } 0 \leq \alpha < 1,\\ \left(x^\alpha h_x\right)(0,t) = 0 & \text{if } 1 \leq \alpha < 2\end{cases} & \text{on } (0,T),\\ h(1,t) = 0 & \text{on } (0,T),\\ h(x,T) = h_T & \text{in } (0,1),\end{cases} \tag{3.8}$$

where $h_T \in L^2_{\mathbb{F}}(\Omega,\mathcal{F}_T,\mathbb{P};L^2(0,1))$, $f_1, x^{-\frac{\alpha}{2}}f_2 \in L^2_{\mathbb{F}}(0,T;L^2(0,1))$.

By the classical well-posedness result for stochastic evolution equations [22], for any $h_T \in L^2_{\mathbb{F}}(\Omega,\mathcal{F}_T,\mathbb{P};L^2(0,1))$, and f_1, $x^{-\frac{\alpha}{2}}f_2 \in L^2_{\mathbb{F}}(0,T;L^2(0,1))$, (3.8) admits a unique solution $(h,H) \in \mathcal{H} \times L^2_{\mathbb{F}}(0,T;L^2(0,1))$.

There also is the following lemma for the backward equation (3.8) from the known result (see Theorem 1.3 in [18]).

LEMMA 3.4 (see [18]). *There exist two positive constants s_4 and λ_4, depending only on α, G_0, and T, such that for any $s \geq s_4$ and $\lambda \geq \lambda_4$, the solution (h,H) to* (3.8) *satisfies*

$$\begin{aligned}&\mathbb{E}\int_Q \theta^2\left(\lambda^3\gamma^3x^{2-\alpha}h^2 + \lambda\gamma x^\alpha h_x^2\right) dxdt\\&\quad\leq C(s)\mathbb{E}\int_Q \theta^2\left(\lambda^3\gamma^3h^2\chi_{G_0} + f_1^2 + \lambda^2\gamma^2x^{-\alpha}f_2^2 + \lambda^2\gamma^2H^2\right) dxdt\end{aligned}$$

for any $h_T \in L^2_{\mathbb{F}}(\Omega,\mathcal{F}_T,\mathbb{P};L^2(0,1))$.

Now, we are in a position to give some estimates for the solution of (3.4) in Case 1 and Case 2, which are the key to proving Theorem 3.2.

Case 1: *$\mathcal{O}_1 = \mathcal{O}_2$ and $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$.* In this case, we denote $\mathcal{O} \triangleq \mathcal{O}_1 = \mathcal{O}_2$ and $\widetilde{\mathcal{O}} \triangleq \widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_2$. From the condition $\mathcal{O} \cap G_0 \neq \emptyset$ in Theorem 2.1, we know that there exists a nonempty open subset ω of $(0,1)$ such that $\omega \subset\subset \mathcal{O} \cap G_0$ and $\omega \cap \widetilde{\mathcal{O}} = \emptyset$.

Put $\eta = \alpha_1 q_1 + \alpha_2 q_2$. Then, (z, Z, η) is the solution of the following coupled equations:

$$\begin{cases} -\mathrm{d}z - \left(x^\alpha z_x\right)_x \mathrm{d}t = \left[a_1 z + \eta\chi_{\mathcal{O}} - (x^{2\alpha}\eta_x)_x\chi_{\widetilde{\mathcal{O}}} + a_2 Z\right]\mathrm{d}t + Z\mathrm{d}W(t) & \text{in } Q,\\ \mathrm{d}\eta - \left(x^\alpha \eta_x\right)_x \mathrm{d}t = \left[a_1\eta - \left(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\right)z\right]\mathrm{d}t + a_2\eta\mathrm{d}W(t) & \text{in } Q,\\ \begin{cases} z(0,t) = \eta(0,t) = 0 & \text{if } 0 \leq \alpha < 1,\\ \left(x^\alpha z_x\right)(0,t) = \left(x^\alpha \eta_x\right)(0,t) = 0 & \text{if } 1 \leq \alpha < 2\end{cases} & \text{on } (0,T),\\ z(1,t) = \eta(1,t) = 0 & \text{on } (0,T),\\ z(x,T) = z_T(x), \quad \eta(x,0) = 0 & \text{in } (0,1).\end{cases} \tag{3.9}$$

From Lemmas 3.3 and 3.4, we have the following result.

PROPOSITION 3.5. *There exist two positive constants λ_5 and s_5, depending only on α, G_0, T, such that for any $\lambda \geq \lambda_5$ and $s \geq s_5$, the solution (z, Z, η) to* (3.9) *satisfies*

$$\begin{aligned}&\mathbb{E}\int_Q \theta^2 \left(\lambda^3\gamma^3 x^{2-\alpha} z^2 + \lambda^5\gamma^5 x^{2-\alpha}\eta^2 + \lambda\gamma x^\alpha z_x^2 + \lambda^3\gamma^3 x^\alpha \eta_x^2\right) dxdt \\ &\leq C\mathbb{E}\int_Q \theta^2\left(\lambda^9\gamma^9\chi_{\mathcal{O}'} z^2 + \lambda^2\gamma^2 Z^2\right) dxdt.\end{aligned} \tag{3.10}$$

Proof. See Appendix B for the proof.

Case 2: $\mathcal{O}_1 \neq \mathcal{O}_2$. In this case, we distinguish two cases for $\mathcal{O}_1$ and $\mathcal{O}_2$. On the one hand, if $\mathcal{O}_1 \cap \mathcal{O}_2 \cap G_0 = \emptyset$, then we can choose two nonempty open sets ω_1 and ω_2 satisfying

$$\omega_i \subset\subset (\mathcal{O}_i \cap G_0), \quad \omega_1 \cap \omega_2 = \emptyset, \tag{3.11}$$

and

$$\overline{\omega}_1 \cap (\mathcal{O}_2 \cap G_0) = \emptyset, \quad \overline{\omega}_2 \cap (\mathcal{O}_1 \cap G_0) = \emptyset. \tag{3.12}$$

This implies that for $i = 1, 2$, there exists a nonempty open subset $\mathcal{O}_i'$ of $(\mathcal{O}_i \cap G_0)$ such that $\overline{\omega_i} \subset \mathcal{O}_i'$, $\mathcal{O}_1' \cap \widetilde{\mathcal{O}}_i = \emptyset$, and $\mathcal{O}_2' \cap \widetilde{\mathcal{O}}_i = \emptyset$. On the other hand, if $(\mathcal{O}_1 \cap G_0) \cap (\mathcal{O}_2 \cap G_0) \neq \emptyset$, then we can choose ω_1 and ω_2 satisfying (3.11), and

$$\overline{\omega}_1 \cap (\mathcal{O}_2 \cap G_0) = \emptyset, \quad \overline{\omega}_2 \subset (\mathcal{O}_1 \cap G_0). \tag{3.13}$$

This implies that there exist two nonempty open subsets $\mathcal{O}_3' \subset ((\mathcal{O}_1 \cap G_0) \setminus \mathcal{O}_2)$ and $\mathcal{O}_4' \subset (\mathcal{O}_1 \cap \mathcal{O}_2 \cap G_0)$ such that $\overline{\omega}_1 \subset \mathcal{O}_3'$, $\mathcal{O}_3' \cap \widetilde{\mathcal{O}}_i = \emptyset$, $\overline{\omega}_2 \subset \mathcal{O}_4'$, and $\mathcal{O}_4' \cap \widetilde{\mathcal{O}}_i = \emptyset$.

Similar to Lemma 3.3, in order to apply Carleman estimates to the solution of (3.4), we need to introduce some weight functions of the form like (3.7). To do so, for $i = 1, 2$, from the condition $\mathcal{O}_i \cap G_0 \neq \emptyset$ in Theorem 2.1, we know that there exists a nonempty open subset $\widetilde{G}_0 \subset\subset G_0$ such that $\mathcal{O}_i \cap \widetilde{G}_0 \neq \emptyset$. Let $d_i(\cdot) \in C^4([0,1])$ be functions satisfying $d_i(x) > 0$ in $(0,1)$, $d_i(0) = d_i(1) = 0$, $d_{i,x}(x) \neq 0$ in $[0,1]\backslash\omega_i'$, and $d_1 = d_2$ in $[0,1]\backslash\widetilde{G}_0$, where $\omega_i' \subseteq \omega_i$. Then, analogously as in (3.7), we can define the following weight functions:

$$\theta_i = e^{-\lambda\gamma(t)\phi_i(x)}, \ \phi_i(x) = \xi(x)\frac{2 - x^{2-\alpha}}{(2-\alpha)^2} + [1 - \xi(x)]\left(e^{2s|d_i|_{C([0,1])}} - e^{sd_i(x)}\right), \tag{3.14}$$

where γ and ξ as in (3.7).

Introduce the following notations:

$$\begin{aligned}&\check{\phi} = \min_{x\in(0,1)}\{\phi_1(x), \phi_2(x)\}, \quad \check{\sigma}(t) = \gamma(t)\check{\phi}, \quad \check{\theta}(t) = e^{-\lambda\check{\sigma}(t)}, \\ &\hat{\phi} = \max_{x\in(0,1)}\{\phi_1(x), \phi_2(x)\}, \quad \hat{\sigma}(t) = \gamma(t)\hat{\phi}, \quad \hat{\theta}(t) = e^{-\lambda\hat{\sigma}(t)}.\end{aligned}$$

Then, we have the following result.

PROPOSITION 3.6. *There exist two positive constants λ_6 and s_6, depending only on α, G_0, T, such that for any $\lambda \geq \lambda_6$ and $s \geq s_6$, the solution (z, Z, q_1, q_2) to* (3.4) *satisfies*

$$\begin{aligned}&\mathbb{E}\int_Q \hat{\theta}^2\left[\lambda^2\gamma^3 x^{2-\alpha} z^2 + \gamma z^2 + \lambda^4\gamma^4 x^{2-\alpha}(q_1^2 + q_2^2) + \lambda^2\gamma^2 x^\alpha (q_{1,x}^2 + q_{2,x}^2)\right] dxdt \\ &\leq C\mathbb{E}\int_Q \check{\theta}^2(\lambda^8\gamma^8 z^2 \chi_{G_0} + \lambda\gamma^2 Z^2) dxdt.\end{aligned} \tag{3.15}$$

Proof. See Appendix C for the proof.

Remark 3.7. Because $(x^{2\alpha}\eta_x)_x\chi_{\widetilde{\mathcal{O}}}$ in the drift term belongs to a Sobolev space of negative order in the first equation of (3.9), we need to use the Carleman estimate by a duality method (Lemma 3.4). The Carleman estimate derived by the weighted identity method is not applicable here.

Remark 3.8. From the proofs of Propositions 3.5 and 3.6, we know that different Carleman estimates lead to different conditions (H_1) and (H_2) on the coefficient a_2, respectively. These conditions mean that a_2 needs to have a bounded weak derivative or suitable singularity.

When dealing with a degenerate problem, a Hardy-type inequality is a very useful tool that will play a crucial role in the proof of Theorem 3.2.

LEMMA 3.9 (Hardy's equality [5]).

(i) *Let* $0 \leq \alpha < 1$. *Then, for all locally absolutely continuous function* y *on* $(0,1)$ *satisfying*

$$\lim_{x\to 0^+} y(x) = 0 \quad \textit{and} \quad \int_0^1 x^\alpha y_x^2 dx < +\infty,$$

the following inequality holds:

$$\int_0^1 x^{\alpha-2} y^2 dx \leq \frac{4}{(1-\alpha)^2}\int_0^1 x^\alpha y_x^2 dx. \tag{3.16}$$

(ii) *Let* $1 < \alpha < 2$. *Then, the above inequality* (3.16) *still holds for all locally absolutely continuous functions* y *on* $(0,1)$ *satisfying*

$$\lim_{x\to 1^-} y(x) = 0 \quad \textit{and} \quad \int_0^1 x^\alpha y_x^2 dx < +\infty.$$

By Propositions 3.5–3.6 and Hardy's equality, we now give the proof of Theorem 3.2.

Proof of Theorem 3.2. We only prove Theorem 3.2 in Case 2, and Case 1 can be proved in a similar way. Let us introduce an auxiliary function $\overline{\psi} \in C^1([0,T])$ with

$$\overline{\psi} = 1 \text{ in } \left[0, \tfrac{T}{2}\right], \quad \overline{\psi} = 0 \text{ in } \left[\tfrac{3T}{4}, T\right], \quad \text{and} \quad |\overline{\psi}_t(t)| \leq \tfrac{C}{T} \text{ in } [0,T],$$

and choose a function $\ell = \ell(t)$ satisfying (2.4). Then, by the first equation of (3.4) and $\mathrm{d}(\overline{\psi} z^2) = \overline{\psi}_t z^2 \mathrm{d}t + 2\overline{\psi} z \mathrm{d}z + \overline{\psi}(\mathrm{d}z)^2$, it follows that

$$\begin{aligned}
&\int_0^1 z^2(0)\mathrm{d}x + \mathbb{E}\int_0^{\frac{T}{2}}\int_0^1 \left(x^\alpha z_x^2 + Z^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad \leq C\mathbb{E}\int_0^{\frac{3T}{4}}\int_0^1 \left(x^\alpha q_{1,x}^2 + x^\alpha q_{2,x}^2 + q_1^2 + q_2^2\right)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 z^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

By Lemma 3.9, it can be deduced that

$$\begin{aligned}
&\int_0^1 z^2(0)\mathrm{d}x + \mathbb{E}\int_0^{\frac{T}{2}}\int_0^1 \left(x^{2-\alpha} z^2 + z^2 + x^\alpha z_x^2 + Z^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad \leq C\mathbb{E}\int_0^{\frac{3T}{4}}\int_0^1 \left(x^\alpha q_{1,x}^2 + x^\alpha q_{2,x}^2 + q_1^2 + q_2^2\right)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 z^2 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Analogously as in (3.7) and (3.14), define the weight functions $\sigma^*(t)=\ell(t)\hat{\phi}$, $\theta^*(t)=e^{-\lambda\sigma^*(t)}$. Then, using the fact that θ^* and ℓ are bounded in $[0,\frac{T}{2}]$, one can get that

$$\begin{aligned}&\int_0^1 z^2(0)\mathrm{d}x+\mathbb{E}\int_0^{\frac{T}{2}}\int_0^1\theta^{*2}\big(\lambda^2\ell^3x^{2-\alpha}z^2+\ell x^\alpha z_x^2+\lambda\ell^2Z^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_0^{\frac{3T}{4}}\int_0^1\big(x^\alpha q_{1,x}^2+x^\alpha q_{2,x}^2+q_1^2+q_2^2\big)\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 z^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.17}$$

By the second equation of (3.4) and Itô's formula, one can obtain that

$$\mathbb{E}\int_0^{\frac{T}{2}}\int_0^1\big(x^\alpha q_{i,x}^2+q_i^2\big)\mathrm{d}x\mathrm{d}t\le C\frac{1}{\mu_i^2}\mathbb{E}\int_0^{\frac{T}{2}}\int_0^1 z^2\mathrm{d}x\mathrm{d}t.$$

By Hardy's inequality, since $\ell(t)=\frac{4^4}{T^8}$ and $(\theta^*)^{-1}=e^{\lambda\ell(t)\hat{\phi}}\le e^{C\lambda}$ in $[0,\frac{T}{2}]$, it follows that

$$\begin{aligned}&\mathbb{E}\int_0^{\frac{T}{2}}\int_0^1\big(x^\alpha q_{1,x}^2+x^\alpha q_{2,x}^2+q_1^2+q_2^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C(\lambda)\left(\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}\right)\mathbb{E}\int_Q\theta^{*2}\big(\lambda^2\ell^3x^{2-\alpha}z^2+\ell x^\alpha z_x^2\big)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

For sufficiently large $\mu_1,\mu_2>0$, the above inequality together with (3.17) implies that

$$\begin{aligned}&\int_0^1 z^2(0)\mathrm{d}x+\mathbb{E}\int_0^{\frac{T}{2}}\int_0^1\theta^{*2}\big(\lambda^2\ell^3x^{2-\alpha}z^2+\ell x^\alpha z_x^2+\lambda\ell^2Z^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1\big(x^\alpha q_{1,x}^2+x^\alpha q_{2,x}^2+q_1^2+q_2^2\big)\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 z^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.18}$$

Since $\theta^*(t)=\hat{\theta}(t)$ in $[\frac{T}{2},T]$ and $\ell(t)=\gamma(t)$ in $[\frac{T}{2},T]$, we can obtain that

$$\mathbb{E}\int_{\frac{T}{2}}^T\int_0^1\theta^{*2}\big(\lambda^2\ell^3x^{2-\alpha}z^2+\lambda\ell^2Z^2\big)\mathrm{d}x\mathrm{d}t\le C\mathbb{E}\int_Q\hat{\theta}^2\big(\lambda^2\gamma^3x^{2-\alpha}z^2+\lambda\ell^2Z^2\big)\mathrm{d}x\mathrm{d}t.$$

Combining the above inequality with (3.15), (3.18), one can find that there is a $\lambda_0>0$ such that for all $\lambda\ge\lambda_0$, it holds that

$$\begin{aligned}&\int_0^1 z^2(0)\mathrm{d}x+\mathbb{E}\int_Q\theta^{*2}\big(\lambda^2\ell^3x^{2-\alpha}z^2+\lambda\ell^2Z^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1\big(x^\alpha q_{1,x}^2+x^\alpha q_{2,x}^2+q_1^2+q_2^2\big)\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 z^2\mathrm{d}x\mathrm{d}t\\&\qquad+C\mathbb{E}\int_Q\hat{\theta}^2\big(\lambda^2\gamma^3x^{2-\alpha}z^2+\lambda\gamma^2Z^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\check{\theta}^2\big(\lambda^8\gamma^8\chi_{G_0}z^2+\lambda\gamma^2Z^2\big)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.19}$$

In what follows, fix $\lambda=\lambda_0$ in (3.19). Set $\widetilde{\rho}(t)=e^{\lambda\sigma^*(t)}$. Then, $\widetilde{\rho}=\widetilde{\rho}(t)$ is a positive nondecreasing function in $C^1([0,1])$ that blows up at $t=T$. By the second equation of (3.4) and Itô's formula, for $i=1,2$ and $t\in[0,T]$,

$$\mathrm{d}\big(\widetilde{\rho}^{-2}q_i^2\big)-\widetilde{\rho}^{-2}(\mathrm{d}q_i)^2=2\widetilde{\rho}^{-2}q_i\left[\big(x^\alpha q_{i,x}\big)_x+a_1q_i-\frac{z\chi_{G_i}}{\mu_i}\right]\mathrm{d}t+2\widetilde{\rho}^{-2}a_2q_i^2\mathrm{d}W(t)+(\widetilde{\rho}^{-2})_tq_i^2\mathrm{d}t.$$

Since $q_i(x,0)=0$ in $[0,1]$ and $(\widetilde{\rho}^{-2})_t \leq 0$ in $[0,T]$, we know that

$$\mathbb{E}\int_0^1 \widetilde{\rho}^{-2}(t) q_i^2(t)\mathrm{d}x + \mathbb{E}\int_0^t\int_0^1 x^\alpha \widetilde{\rho}^{-2} q_{i,x}^2 \mathrm{d}x\mathrm{d}s \leq C\mathbb{E}\int_0^t\int_0^1 \widetilde{\rho}^{-2}\left(q_i^2 + \frac{z^2}{\mu_i^2}\chi_{G_i}\right)\mathrm{d}x\mathrm{d}s.$$

From Gronwall's inequality and $0\notin G_i (i=1,2)$, it follows that

$$\mathbb{E}\int_Q \widetilde{\rho}^{-2}\big(q_i^2 + x^\alpha q_{i,x}^2\big)\mathrm{d}x\mathrm{d}t \leq \frac{C}{\mu_i^2}\mathbb{E}\int_Q \widetilde{\rho}^{-2} z^2 \chi_{G_i}\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \widetilde{\rho}^{-2} x^{2-\alpha} z^2 \mathrm{d}x\mathrm{d}t.$$

Noting that $\widetilde{\rho}^{-2} = \theta^{*2}$, by the above estimate and (3.19), one can get

$$\begin{aligned}
&\int_0^1 z^2(0)\mathrm{d}x + \sum_{i=1}^2 \mathbb{E}\int_Q \widetilde{\rho}^{-2}\big(q_i^2 + x^\alpha q_{i,x}^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad\leq \int_0^1 z^2(0)\mathrm{d}x + C\mathbb{E}\int_Q \theta^{*2} x^{2-\alpha} z^2\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \check{\theta}^2\big(\lambda_0^8\gamma^8 z^2\chi_{G_0} + \lambda_0\gamma^2 Z^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

This gives the desired estimate (3.5). ∎

4. Inverse problem. This section is devoted to studying the inverse problem of (1.1) under a Nash equilibrium strategy. We consider the inverse problem for the following stochastic degenerate parabolic equation:

$$\begin{cases}
\mathrm{d}y - \big(x^\alpha y_x\big)_x \mathrm{d}t = (a_1 y + v_1\chi_{G_1} + v_2\chi_{G_2})\mathrm{d}t + a_2 y\mathrm{d}W(t) & \text{in } Q,\\
\begin{cases} y(0,t)=0 & \text{if } 0\leq\alpha<1,\\ \big(x^\alpha y_x\big)(0,t)=0 & \text{if } 1\leq\alpha<2\end{cases} & \text{on } (0,T),\\
y(1,t)=0 & \text{on } (0,T),\\
y(x,0)=y_0(x) & \text{in } (0,1),
\end{cases}$$

where $y_0\in L^2(0,1)$. From the conclusion in Theorem 3.1, the pair $(\overline{v}_1,\overline{v}_2)$ is a Nash equilibrium if and only if

$$\overline{v}_i = -\frac{1}{\mu_i} p_i \chi_{G_i}, \quad i=1,2,$$

where (y,p_i,P_i) is the solution of the coupled stochastic forward-backward equations,

$$\begin{cases}
\mathrm{d}y - \big(x^\alpha y_x\big)_x\mathrm{d}t = (a_1 y - \frac{p_1}{\mu_1}\chi_{G_1} - \frac{p_2}{\mu_2}\chi_{G_2})\mathrm{d}t + a_2 y\mathrm{d}W(t) & \text{in } Q,\\
-\mathrm{d}p_i - \big(x^\alpha p_{i,x}\big)_x\mathrm{d}t = \big[a_1 p_i + a_2 P_i + \alpha_i(y-y_i^*)\chi_{\mathcal{O}_i} & \\
\qquad\qquad - \alpha_i(x^{2\alpha}y_x - x^\alpha y_{i+2}^*)_x\chi_{\widetilde{\mathcal{O}}_i}\big]\mathrm{d}t + P_i\mathrm{d}W(t) & \text{in } Q,\\
\begin{cases} y(0,t)=p_i(0,t)=0 & \text{if } 0\leq\alpha<1,\\ \big(x^\alpha y_x\big)(0,t)=\big(x^\alpha p_{i,x}\big)(0,t)=0 & \text{if } 1\leq\alpha<2\end{cases} & \text{on } (0,T),\\
y(1,t)=p_i(1,t)=0 & \text{on } (0,T),\\
y(x,0)=y_0(x),\quad p_i(x,T)=0 & \text{in } (0,1).
\end{cases} \tag{4.1}$$

In this part, in order to study the inverse initial value problem, we get the interpolation inequality (2.5) for the solution of (4.1) by some Carleman estimates. Carleman estimates are a class of energy estimates with exponential weights, which can be used to study the uniqueness, controllability, and inverse problem of the deterministic and stochastic partial differential equations. We refer to [7, 10, 12, 17, 21, 30, 35] in this respect.

First, we establish the following global Carleman estimate for the forward stochastic degenerate parabolic equation (3.6).

THEOREM 4.1. *There exist two positive constants λ_1 and s_1 such that for all $\lambda \geq \lambda_1$ and $s \geq s_1$, the solution y to* (3.6) *satisfies*

$$\begin{aligned}\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 y^2 + \lambda x^\alpha y_x^2 + s\lambda g^2\big)dxdt \\ \leq C\left[\mathbb{E}\int_0^1 \left(\widetilde{\theta}^2(T)s\lambda y^2(T) + \widetilde{\theta}^2(0)x^\alpha y_x^2(0)\right)dx + \mathbb{E}\int_Q \widetilde{\theta}^2\big(f^2 + x^\alpha g_x^2\big)dxdt\right],\end{aligned} \tag{4.2}$$

where $\widetilde{\theta} = e^{s\varphi}$, $\varphi = e^{\lambda\widetilde{\psi}}$, and $\widetilde{\psi}(t) = t$.

Proof. See Appendix D for the proof.

Next, we establish Carleman estimates for the backward stochastic degenerate parabolic equation (3.8) by two different methods. Based on a weighted identity for the backward stochastic degenerate parabolic operator, one has the following Carleman estimate with diffusion terms in $L^2(G)$-space for (3.8).

THEOREM 4.2. *Assume that $f_2 = 0$ and $h_T = 0$. Then, there exist two positive constants λ_2 and s_2 such that for any $\lambda \geq \lambda_2$ and $s \geq s_2$, the solution (h, H) to* (3.8) *satisfies*

$$\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2 + x^\alpha H_x^2\big)dxdt \leq C\mathbb{E}\int_Q \widetilde{\theta}^2 f_1^2 dxdt. \tag{4.3}$$

The proof of Theorem 4.2 is similar to that of Theorem 4.1 and thus is omitted here. In fact, the proofs of both Theorems 4.1 and 4.2 are similar to that of Theorem 1.1 in [20].

By a duality technique and Theorem 4.2, another Carleman estimate for (3.8) can be established with diffusion terms in a Sobolev space of negative order.

THEOREM 4.3. *Assume that $h_T = 0$. Then, there exist two positive constants λ_3 and s_3 such that for any $\lambda \geq \lambda_3$ and $s \geq s_3$, the solution (h, H) to* (3.8) *satisfies*

$$\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2\big)dxdt \leq C\mathbb{E}\int_Q \widetilde{\theta}^2\big(f_1^2 + s\lambda x^\alpha f_2^2\big)dxdt. \tag{4.4}$$

Proof. See Appendix E for the proof.

Remark 4.4. The Carleman estimate (4.4) is different from the result in [18]. On the one hand, the weight function $\widetilde{\theta}$ is independent of the space variable, and the terminal data is needed to be 0 in (3.8). On the other hand, there is no local information about the solution and the diffusion term on the right-hand side of (4.4).

Now, by Carleman estimates (4.2) and (4.4), we are in position to prove Theorem 2.6.

Proof of Theorem 2.6. Choose a cut-off function $\varsigma \in C^\infty([0,T];[0,1])$ such that $\varsigma(t) = 1$ in $[t_1, T]$, and $\varsigma(t) = 0$ in $[0, t_2]$, where $0 < t_2 < t_1 < t_0$. Set $z = \varsigma y$, $r_i = \varsigma p_i$, and $R_i = \varsigma P_i$ $(i = 1, 2)$. Then, by means of (y, p_i, P_i) solving (4.1), we know that (z, r_i, R_i) is the solution of the following equations:

$$
\left\{
\begin{array}{lll}
\mathrm{d}z - \big(x^\alpha z_x\big)_x \mathrm{d}t = (a_1 z - \frac{1}{\mu_1} r_1 \chi_{G_1} - \frac{1}{\mu_2} r_2 \chi_{G_2} + \varsigma_t y)\mathrm{d}t & & \\
\qquad\qquad + a_2 z \mathrm{d}W(t) & & \text{in } Q, \\
-\mathrm{d}r_i - \big(x^\alpha r_{i,x}\big)_x \mathrm{d}t = E_1(r_i, R_i, y)\mathrm{d}t + R_i \mathrm{d}W(t) & & \text{in } Q, \\
\left\{ \begin{array}{ll} z(0,t) = r_i(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ \big(x^\alpha z_x\big)(0,t) = \big(x^\alpha r_{i,x}\big)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{array} \right. & & \text{on } (0,T), \\
z(1,t) = r_i(1,t) = 0 & & \text{on } (0,T), \\
z(x,0) = 0, \quad r_i(x,T) = 0 & & \text{in } (0,1),
\end{array}
\right. \tag{4.5}
$$

where $E_1(r_i,R_i,p_i,y) = a_1 r_i + a_2 R_i + \varsigma_t p_i + \varsigma\alpha_i(y - y_i^*)\chi_{\mathcal{O}_i} - \varsigma\alpha_i(x^{2\alpha} y_x - x^\alpha y_{i+2}^*)_x \chi_{\widetilde{\mathcal{O}}_i}$. Applying Theorem 4.1 to the first equation of (4.5) gives

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 z^2 + \lambda x^\alpha z_x^2\big)\mathrm{d}x\mathrm{d}t \\
&\quad \le C\mathbb{E}\int_Q \widetilde{\theta}^2\big(z^2 + r_1^2\chi_{G_1} + r_2^2\chi_{G_2} + \varsigma_t^2 y^2 + x^\alpha z_x^2\big)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^1 s\lambda\widetilde{\theta}^2(T) z^2(T)\mathrm{d}x.
\end{aligned} \tag{4.6}
$$

Applying Theorem 4.3 to the second equation of (4.5), it holds that

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^2\big[s\lambda^2(r_1^2 + r_2^2) + \lambda x^\alpha(r_{1,x}^2 + r_{2,x}^2) + s\lambda(R_1^2 + R_2^2)\big]\mathrm{d}x\mathrm{d}t \\
&\quad \le C\mathbb{E}\int_Q \widetilde{\theta}^2\big[r_1^2 + r_2^2 + R_1^2 + R_2^2 + \varsigma_t^2(p_1^2 + p_2^2) + \varsigma^2\big(y^2 + y_1^{*2}\chi_{\mathcal{O}_1} + y_2^{*2}\chi_{\mathcal{O}_2}\big) \\
&\qquad + s\lambda x^\alpha \varsigma^2\big(x^{4\alpha} y_x^2 + x^{2\alpha} y_3^{*2}\chi_{\widetilde{\mathcal{O}}_1} + x^{2\alpha} y_4^{*2}\chi_{\widetilde{\mathcal{O}}_2}\big)\big]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.7}
$$

By the definitions of ς and $\widetilde{\theta}$, one can see that

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^2\varsigma_t^2\big(s\lambda y^2 + p_1^2 + p_2^2\big)\mathrm{d}x\mathrm{d}t \\
&\quad \le C\mathbb{E}\int_{t_2}^{t_1}\int_0^1 \widetilde{\theta}^2\big(s\lambda y^2 + p_1^2 + p_2^2\big)\mathrm{d}x\mathrm{d}t \le C\widetilde{\theta}^2(t_1)\mathbb{E}\int_Q \big(s\lambda y^2 + p_1^2 + p_2^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This together with (4.6) and (4.7) implies that for a sufficiently large $\lambda > 0$,

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^2\varsigma^2\big[s\lambda^2(s\lambda y^2 + p_1^2 + p_2^2) + \lambda x^\alpha(s\lambda y_x^2 + p_{1,x}^2 + p_{2,x}^2) + s\lambda(P_1^2 + P_2^2)\big]\mathrm{d}x\mathrm{d}t \\
&\quad = \mathbb{E}\int_Q \widetilde{\theta}^2\big[s\lambda^2(s\lambda z^2 + r_1^2 + r_2^2) + \lambda x^\alpha(s\lambda z_x^2 + r_{1,x}^2 + r_{2,x}^2) + s\lambda(R_1^2 + R_2^2)\big]\mathrm{d}x\mathrm{d}t \\
&\quad \le C\widetilde{\theta}^2(t_1)\mathbb{E}\int_Q \big(s\lambda y^2 + p_1^2 + p_2^2\big)\mathrm{d}x\mathrm{d}t + C\widetilde{\theta}^2(T)\left(\mathbb{E}\int_0^1 s^2\lambda^2 y^2(T)\mathrm{d}x + M\right),
\end{aligned}
$$

where $M = \mathbb{E}\int_Q (y_1^{*2}\chi_{\mathcal{O}_1} + y_2^{*2}\chi_{\mathcal{O}_2} + x^{2\alpha} y_3^{*2}\chi_{\widetilde{\mathcal{O}}_1} + x^{2\alpha} y_4^{*2}\chi_{\widetilde{\mathcal{O}}_2})\mathrm{d}x\mathrm{d}t$. From the above inequality, one can see that

$$
\begin{aligned}
&\mathbb{E}\int_{t_0}^T\int_0^1 \big[s\lambda^2(s\lambda y^2 + p_1^2 + p_2^2) + \lambda x^\alpha(s\lambda y_x^2 + p_{1,x}^2 + p_{2,x}^2) + s\lambda(P_1^2 + P_2^2)\big]\mathrm{d}x\mathrm{d}t \\
&\quad \le C\widetilde{\theta}^{-2}(t_0)\widetilde{\theta}^2(t_1)\mathbb{E}\int_Q \big(s\lambda y^2 + p_1^2 + p_2^2\big)\mathrm{d}x\mathrm{d}t + C\widetilde{\theta}^2(T)\left(\mathbb{E}\int_0^1 s^2\lambda^2 y^2(T)\mathrm{d}x + M\right).
\end{aligned}
$$

By means of $\mathrm{d}(y^2) = 2y\mathrm{d}y + (\mathrm{d}y)^2$ and the first equation of (4.1), we obtain

$$\begin{aligned}\mathbb{E}\int_0^1 y^2(t_0)\mathrm{d}x &= \mathbb{E}\int_0^1 y^2(T)\mathrm{d}x - \mathbb{E}\int_{t_0}^T\int_0^1 \left[2y\mathrm{d}y + (\mathrm{d}y)^2\right]\mathrm{d}x \\ &\le \mathbb{E}\int_0^1 y^2(T)\mathrm{d}x + C\mathbb{E}\int_{t_0}^T\int_0^1 \left(x^\alpha y_x^2 + y^2 + p_1^2\chi_{G_1} + p_2^2\chi_{G_2}\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then, for any $s > 0$, it holds that

$$\mathbb{E}\int_0^1 y^2(t_0)\mathrm{d}x \le C\frac{\widetilde{\theta}^2(t_1)}{\widetilde{\theta}^2(t_0)}\mathbb{E}\int_Q \left(s\lambda y^2 + p_1^2 + p_2^2\right)\mathrm{d}x\mathrm{d}t + C\widetilde{\theta}^2(T)\left(\mathbb{E}\int_0^1 s^2\lambda^2 y^2(T)\mathrm{d}x + M\right).$$

Now, fixing $\lambda = \lambda_0$, from the above inequality, we have

$$\mathbb{E}\int_0^1 y^2(t_0)\mathrm{d}x \le C\mathbb{E}\int_Q e^{2s(e^{\lambda t_1} - e^{\lambda t_0})}\left(y^2 + p_1^2 + p_2^2\right)\mathrm{d}x\mathrm{d}t + Ce^{2se^{\lambda T}}\left(\mathbb{E}\int_0^1 y^2(T)\mathrm{d}x + M\right).$$

Choosing an $s > 1$ as a minimizer of the right-hand side in the above inequality. Then, we get

$$\mathbb{E}\int_0^1 y^2(t_0)\mathrm{d}x \le C\left[\mathbb{E}\int_Q \left(y^2 + p_1^2 + p_2^2\right)\mathrm{d}x\mathrm{d}t\right]^\kappa \left(\mathbb{E}\int_0^1 y^2(T)\mathrm{d}x + M\right)^{1-\kappa},$$

where $\kappa = \frac{e^{\lambda T}}{e^{\lambda T} + (e^{\lambda t_1} - e^{\lambda t_0})}$. This completes the proof of Theorem 2.6. ∎

5. Summary. This paper considers controllability and the inverse initial problem for stochastic degenerate parabolic equations under a Nash equilibrium strategy, which are combinations of game theory with controllability and the inverse problem, respectively. The characterization of the Nash equilibrium is given, and the origin equation (1.1) can be transformed into the coupled stochastic forward-backward equations (3.3). By Carleman estimates, the hierarchical controllability of the coupled stochastic equations is proved, and the uniqueness of an inverse initial value problem is obtained under the condition of terminal observation and the given goals.

It will be interesting to further consider the following problems:

(i) the optimal control problems with endpoint/state constraints for the Nash equilibrium,

(ii) the Stackelberg–Nash type controllability problem for stochastic hyperbolic equations,

(iii) the inverse source problem for stochastic partial differential equations under a Nash equilibrium strategy.

Appendix A. Proof of Theorem 3.1. We use the Lax–Milgram theorem [4] to prove the existence and uniqueness of a Nash equilibrium. For $i = 1, 2$, introduce the spaces $\mathcal{U}_i = L^2_{\mathbb{F}}(0,T;L^2(G_i)), U_2 = \mathcal{U}_1 \times \mathcal{U}_2$. Define the operators $I_i, L_i \in \mathcal{L}(\mathcal{U}_i; L^2_{\mathbb{F}}(0,T;L^2(0,1)))$ $(i = 1,2)$ as

$$L_i v_i = \widetilde{y}_i \quad \text{and} \quad I_i v_i = x^\alpha \widetilde{y}_{i,x},$$

where $\widetilde{y}_i$ is the solution of the following equation:

$$\text{(A.1)}\qquad \begin{cases} \mathrm{d}\widetilde{y}_i - \left(x^\alpha \widetilde{y}_{i,x}\right)_x \mathrm{d}t = (a_1\widetilde{y}_i + v_i\chi_{G_i})\mathrm{d}t + a_2\widetilde{y}_i\mathrm{d}W(t) & \text{in } Q, \\ \begin{cases} \widetilde{y}_i(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ \left(x^\alpha \widetilde{y}_{i,x}\right)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{cases} & \text{on } (0,T), \\ \widetilde{y}_i(1,t) = 0 & \text{on } (0,T), \\ \widetilde{y}_i(x,0) = 0 & \text{in } (0,1). \end{cases}$$

Then, the solution y to (1.1) can be written as $y=L_1\overline{v}_1+L_2\overline{v}_2+\beta$, where β solves

$$\begin{cases} \mathrm{d}\beta-\big(x^\alpha\beta_x\big)_x\mathrm{d}t=(a_1\beta+u_1\chi_{G_0})\mathrm{d}t+(a_2\beta+u_2)\mathrm{d}W(t) & \text{in } Q,\\ \begin{cases}\beta(0,t)=0 & \text{if } 0\le\alpha<1,\\ \big(x^\alpha\beta_x\big)(0,t)=0 & \text{if } 1\le\alpha<2\end{cases} & \text{on } (0,T),\\ \beta(1,t)=0 & \text{on } (0,T),\\ \beta(x,0)=y_0(x) & \text{in } (0,1).\end{cases}$$

By the definitions of J_1 and J_2, the pair $(\overline{v}_1,\overline{v}_2)\in U_2$ is the Nash equilibrium if and only if, for any $v_i\in\mathcal{U}_i$, $i=1,2$,

$$\text{(A.2)}\qquad \mathbb{E}\int_Q\big[\alpha_i(y-y_i^*)\widetilde{y}_i\chi_{\mathcal{O}_i}+\alpha_i(x^\alpha y_x-y_{i+2}^*)x^\alpha\widetilde{y}_{i,x}\chi_{\widetilde{\mathcal{O}}_i}+\mu_iv_i\overline{v}_i\chi_{G_i}\big]\mathrm{d}x\mathrm{d}t=0,$$

that is,

$$\begin{aligned}\mathbb{E}\int_Q\big\{&\alpha_i\big[L_1\overline{v}_1+L_2\overline{v}_2-(y_i^*-\beta)\big]L_iv_i\chi_{\mathcal{O}_i}+\alpha_i\big[I_1\overline{v}_1+I_2\overline{v}_2\\ &-(y_{i+2}^*-x^\alpha\beta_x)\big]I_iv_i\chi_{\widetilde{\mathcal{O}}_i}\big\}\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_{G_i}\mu_iv_i\overline{v}_i\mathrm{d}x\mathrm{d}t=0.\end{aligned}$$

Hence, for all $v_i\in\mathcal{U}_i$, $i=1,2$,

$$\begin{aligned}&\alpha_i\big(L_i^*((L_1\overline{v}_1+L_2\overline{v}_2-(y_i^*-\beta))\chi_{\mathcal{O}_i}),v_i\big)\\ &\quad+\alpha_i\big(I_i^*((I_1\overline{v}_1+I_2\overline{v}_2-(y_{i+2}^*-x^\alpha\beta_x))\chi_{\widetilde{\mathcal{O}}_i}),v_i\big)+\mu_i\big(\overline{v}_i\chi_{G_i},v_i\big)=0,\end{aligned}$$

where $L_i^*,I_i^*\in\mathcal{L}(L^2_{\mathbb{F}}(0,T;L^2(0,1));\mathcal{U}_i)$ are the adjoint operators of L_i,I_i, respectively. This implies that $(\overline{v}_1,\overline{v}_2)$ is a Nash equilibrium if and only if

$$\begin{aligned}&\alpha_iL_i^*((L_1\overline{v}_1+L_2\overline{v}_2)\chi_{\mathcal{O}_i})+\alpha_iI_i^*((I_1\overline{v}_1+I_2\overline{v}_2)\chi_{\widetilde{\mathcal{O}}_i})+\mu_i\overline{v}_i\\ &\quad=\alpha_iL_i^*((y_i^*-\beta)\chi_{\mathcal{O}_i})+\alpha_iI_i^*((y_{i+2}^*-x^\alpha\beta_x)\chi_{\widetilde{\mathcal{O}}_i}),\ \ i=1,2.\end{aligned}$$

Let us introduce the operator $M\in\mathcal{L}(U_2;U_2)$, with

$$\begin{aligned}M(v_1,v_2)=\big(&\alpha_1L_1^*((L_1v_1+L_2v_2)\chi_{\mathcal{O}_1})+\alpha_1I_1^*((I_1v_1+I_2v_2)\chi_{\widetilde{\mathcal{O}}_1})+\mu_1v_1,\\ &\alpha_2L_2^*((L_1v_1+L_2v_2)\chi_{\mathcal{O}_2})+\alpha_2I_2^*((I_1v_1+I_2v_2)\chi_{\widetilde{\mathcal{O}}_2})+\mu_2v_2\big),\end{aligned}$$

for all $(v_1,v_2)\in U_2$, and define the functional $b:U_2\times U_2\to\mathbb{R}$ as

$$b\big((v_1,v_2),(\widetilde{v}_1,\widetilde{v}_2)\big)=\big(M(v_1,v_2),(\widetilde{v}_1,\widetilde{v}_2)\big)_{U_2}\ \ \forall\,(v_1,v_2),(\widetilde{v}_1,\widetilde{v}_2)\in U_2.$$

Then, one can see that b is bounded and coercive. Applying the Lax–Milgram theorem [4], there is a unique solution $(v_1,v_2)\in U_2$ satisfying the equation $M(v_1,v_2)=\Psi$, where

$$\begin{aligned}\Psi=\big(&\alpha_1L_1^*((y_1^*-\beta)\chi_{\mathcal{O}_1})+\alpha_1I_1^*((y_3^*-x^\alpha\beta_x)\chi_{\widetilde{\mathcal{O}}_1}),\\ &\alpha_2L_2^*((y_2^*-\beta)\chi_{\mathcal{O}_2})+\alpha_2I_2^*((y_4^*-x^\alpha\beta_x)\chi_{\widetilde{\mathcal{O}}_2})\big)\in U_2.\end{aligned}$$

Next, we give a characterization of a Nash equilibrium. Introduce the following backward stochastic degenerate parabolic equation which is the adjoint equation of (A.1):

$$
(\mathrm{A.3})\qquad \begin{cases} -\mathrm{d}p_i - \big(x^\alpha p_{i,x}\big)_x \mathrm{d}t = \big[a_1 p_i + a_2 P_i + \alpha_i (y - y_i^*)\chi_{\mathcal{O}_i} \\ \qquad\qquad - \alpha_i (x^{2\alpha} y_x - x^\alpha y_{i+2}^*)_x \chi_{\widetilde{\mathcal{O}}_i}\big]\mathrm{d}t + P_i \mathrm{d}W(t) & \text{in } Q, \\ \begin{cases} p_i(0,t) = 0 & \text{if } 0 \le \alpha < 1, \\ \big(x^\alpha p_{i,x}\big)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{cases} & \text{on } (0,T), \\ p_i(1,t) = 0 & \text{on } (0,T), \\ p_i(x,T) = 0 & \text{in } (0,1). \end{cases}
$$

From (A.1) and (A.3), we have

$$
\alpha_i \mathbb{E}\int_Q \big[(y - y_i^*)\widetilde{y}_i \chi_{\mathcal{O}_i} + (x^\alpha y_x - y_{i+2}^*) x^\alpha \widetilde{y}_{i,x} \chi_{\widetilde{\mathcal{O}}_i}\big]\mathrm{d}x\mathrm{d}t = \mathbb{E}\int_0^T \int_{G_i} p_i v_i \mathrm{d}x\mathrm{d}t.
$$

Combining the above inequality with (A.2) gives

$$
\mathbb{E}\int_0^T \int_{G_i} \big(p_i + \mu_i \overline{v}_i\big) v_i \mathrm{d}x\mathrm{d}t = 0 \quad \forall v_i \in \mathcal{U}_i.
$$

This implies the desired results (3.2) and (3.3). ◻

Appendix B. Proof of Proposition 3.5. Applying Lemma 3.4 to the first equation of (3.9) leads to

$$
(\mathrm{B.1})\qquad \begin{aligned} &\mathbb{E}\int_Q \theta^2 (\lambda^3\gamma^3 x^{2-\alpha} z^2 + \lambda\gamma x^\alpha z_x^2)\mathrm{d}x\mathrm{d}t \\ &\quad \le C\mathbb{E}\int_Q \theta^2 \big[\lambda^3\gamma^3 \chi_\omega z^2 + z^2 + \eta^2 \chi_{\mathcal{O}} + \lambda^2\gamma^2 x^{-\alpha} (x^{2\alpha}\eta_x \chi_{\widetilde{\mathcal{O}}})^2 + \lambda^2\gamma^2 Z^2\big]\mathrm{d}x\mathrm{d}t. \end{aligned}
$$

By Hardy's inequality, when $\alpha \ne 1$,

$$
(\mathrm{B.2})\qquad \mathbb{E}\int_Q \theta^2 z^2 \mathrm{d}x\mathrm{d}t \le \mathbb{E}\int_Q \theta^2 x^{\alpha-2} z^2 \mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta^2 (x^\alpha z_x^2 + \lambda^2\gamma^2 x^{2-\alpha} z^2)\mathrm{d}x\mathrm{d}t,
$$

and when $\alpha = 1$,

$$
(\mathrm{B.3})\qquad \begin{aligned} \mathbb{E}\int_Q \theta^2 z^2 \mathrm{d}x\mathrm{d}t &\le \mathbb{E}\int_Q \theta^2 x^{-\frac{1}{3}} z^2 \mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q x^{\frac{5}{3}} (\theta z)_x^2 \mathrm{d}x\mathrm{d}t \\ &\le C\mathbb{E}\int_Q x^\alpha (\theta z)_x^2 \mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta^2 (x^\alpha z_x^2 + \lambda^2\gamma^2 x^{2-\alpha} z^2)\mathrm{d}x\mathrm{d}t. \end{aligned}
$$

Combining (B.1)–(B.3) indicates that for a sufficiently large $\lambda > 0$,

$$
(\mathrm{B.4})\qquad \begin{aligned} &\mathbb{E}\int_Q \theta^2 (\lambda^3\gamma^3 x^{2-\alpha} z^2 + \lambda\gamma x^\alpha z_x^2)\mathrm{d}x\mathrm{d}t \\ &\quad \le C\mathbb{E}\int_Q \theta^2 \big(\lambda^3\gamma^3 z^2 \chi_\omega + \eta^2 \chi_{\mathcal{O}} + \lambda^2\gamma^2 x^{3\alpha} \eta_x^2 \chi_{\widetilde{\mathcal{O}}} + \lambda^2\gamma^2 Z^2\big)\mathrm{d}x\mathrm{d}t. \end{aligned}
$$

We can prove the inequality (3.10) under the condition (H_2) by Lemma 3.4. Without loss of generality, we assume that (H_1) holds. Applying Lemma 3.3 to the second equation of (3.9), one can find that

$$
(\mathrm{B.5})\qquad \begin{aligned} &\mathbb{E}\int_Q \theta^2 \lambda^3\gamma^3 \big(\lambda^2\gamma^2 x^{2-\alpha} \eta^2 + x^\alpha \eta_x^2\big)\mathrm{d}x\mathrm{d}t \\ &\quad \le C\mathbb{E}\int_Q \theta^2 \lambda^2\gamma^2 \left[\lambda^3\gamma^3 \eta^2 \chi_\omega + \left(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\right)^2 z^2\right]\mathrm{d}x\mathrm{d}t. \end{aligned}
$$

Because $G_i (i = 1, 2)$ are the control areas, we can assume that $0 \notin G_i (i = 1, 2)$. By (B.4), (B.5), and Hardy's inequality, it follows that for sufficiently large $\lambda > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q \theta^2\big(\lambda^3\gamma^3 x^{2-\alpha}z^2 + \lambda^5\gamma^5 x^{2-\alpha}\eta^2 + \lambda\gamma x^\alpha z_x^2 + \lambda^3\gamma^3 x^\alpha \eta_x^2\big)\mathrm{d}x\mathrm{d}t \\ &\quad\le C\mathbb{E}\int_Q \theta^2\big(\lambda^3\gamma^3 z^2\chi_\omega + \lambda^5\gamma^5\eta^2\chi_\omega + \lambda^2\gamma^2 Z^2\big)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{B.6}$$

In the following, we will give an estimate of $\mathbb{E}\int_Q \theta^2\lambda^5\gamma^5\chi_\omega\eta^2\mathrm{d}x\mathrm{d}t$. To do so, choose a function $\zeta \in C_0^\infty(\mathbb{R})$ satisfying that $0 \le \zeta(x) \le 1$, $\mathrm{supp}\zeta \subseteq \mathcal{O}'$, $\zeta(x) = 1$ in ω, $|\frac{\zeta_x}{\zeta^{\frac{1}{2}}}| \le C$, $|\frac{\zeta_{xx}}{\zeta^{\frac{1}{2}}}| \le C$, where $\mathcal{O}'$ is an open set satisfying $\overline{\omega} \subseteq \mathcal{O}'$, $\overline{\mathcal{O}'} \subseteq \mathcal{O} \cap G_0$, and $\mathcal{O}' \cap \widetilde{\mathcal{O}} = \emptyset$. Then, by Itô's formula, we have

$$\begin{aligned} 0 &= \mathbb{E}\int_Q \mathrm{d}(\zeta\theta^2\lambda^5\gamma^5\eta z)\mathrm{d}x \\ &= \mathbb{E}\int_Q \zeta\theta^2\lambda^5\gamma^5\Big\{\eta\Big[-(x^\alpha z_x)_x - a_1 z - \eta\chi_{\mathcal{O}} + \big(x^{2\alpha}\eta_x\big)_x\chi_{\widetilde{\mathcal{O}}} - a_2 Z\Big] + a_2\eta Z \\ &\quad + z\Big[\big(x^\alpha\eta_x\big)_x + a_1\eta - \Big(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\Big)z\Big]\Big\}\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \big(\zeta\theta^2\lambda^5\gamma^5\big)_t\eta z\mathrm{d}x\mathrm{d}t \\ &= \mathbb{E}\int_Q \Big[\big(\zeta\theta^2\lambda^5\gamma^5\big)_t\eta z - \big(\zeta\theta^2\lambda^5\gamma^5\big)_{xx}\eta x^\alpha z - \big(\zeta\theta^2\lambda^5\gamma^5\big)_x\big(2\eta_x x^\alpha z + \eta(x^\alpha)_x z\big) \\ &\quad - \zeta\theta^2\lambda^5\gamma^5\eta^2 - \zeta\theta^2\lambda^5\gamma^5\left(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\right)z^2\Big]\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{B.7}$$

Hence, by Hölder's inequality and Hardy's inequality, for any $\epsilon > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q \zeta\theta^2\lambda^5\gamma^5\Big[\eta^2 + \Big(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\Big)z^2\Big]\mathrm{d}x\mathrm{d}t \\ &\quad = \mathbb{E}\int_Q \Big[\big(\zeta\theta^2\lambda^5\gamma^5\big)_t\eta - \big(\zeta\theta^2\lambda^5\gamma^5\big)_{xx}\eta x^\alpha - \big(\zeta\theta^2\lambda^5\gamma^5\big)_x\big(2\eta_x x^\alpha + \eta(x^\alpha)_x\big)\Big]z\mathrm{d}x\mathrm{d}t \\ &\quad \le C\mathbb{E}\int_Q \theta^2\Big[\zeta\big(\lambda^7\gamma^7|\eta x^\alpha z| + \lambda^6\gamma^6|\eta x^{\alpha-1}z| + \lambda^6\gamma^6|z x^\alpha\eta_x| + \lambda^6\gamma^7|\eta z|\big) \\ &\qquad + |\zeta_x|\big(\lambda^6\gamma^6|\eta x^\alpha z| + \lambda^5\gamma^5|\eta x^{\alpha-1}z| + \lambda^5\gamma^5|z x^\alpha\eta_x|\big) + |\zeta_{xx}|\lambda^5\gamma^5|\eta x^\alpha z|\Big]\mathrm{d}x\mathrm{d}t \\ &\quad \le C\mathbb{E}\int_Q \theta^2\left[\zeta\big(\epsilon\lambda^5\gamma^5\eta^2 + \frac{1}{\epsilon}\lambda^9\gamma^9 z^2 + \epsilon\lambda^5\gamma^5 x^{2-\alpha}\eta^2 + \epsilon\lambda^3\gamma^3 x^\alpha\eta_x^2\big)\right. \\ &\qquad \left. + \epsilon\frac{\zeta_x^2}{\zeta}\lambda^3\gamma^3 x^\alpha\eta^2 + \frac{\zeta_x^2}{\epsilon\zeta}\lambda^9\gamma^9 z^2 + \frac{\zeta_{xx}^2}{\epsilon\zeta}\lambda^9\gamma^9 z^2\right]\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Combining the above inequality with (B.6) gives the desired inequality. □

Appendix C. Proof of Proposition 3.6. Choose a function $\Lambda \in C^4([0,1])$ such that $0 \le \Lambda(x) \le 1$ in $(0,1)$, $\Lambda(x) = 0$ in $\widetilde{G}_0$, and $\Lambda(x) = 1$ in $(0,1) \setminus G_0$. Let $\widetilde{z} = \Lambda z$ and $\widetilde{Z} = \Lambda Z$. Then, $(\widetilde{z}, \widetilde{Z})$ is the solution of the following equation:

$$\begin{aligned} -\mathrm{d}\widetilde{z} - \big(x^\alpha\widetilde{z}_x\big)_x\mathrm{d}t &= \big[a_1\widetilde{z} + \alpha_1 q_1\Lambda\chi_{\mathcal{O}_1} + \alpha_2 q_2\Lambda\chi_{\mathcal{O}_2} - \big(\alpha_1\Lambda x^{2\alpha}q_{1,x}\big)_x\chi_{\widetilde{\mathcal{O}}_1} \\ &\quad + \alpha_1 x^{2\alpha}\Lambda_x q_{1,x}\chi_{\widetilde{\mathcal{O}}_1} - \big(\alpha_2\Lambda x^{2\alpha}q_{2,x}\big)_x\chi_{\widetilde{\mathcal{O}}_2} + \alpha_2 x^{2\alpha}\Lambda_x q_{2,x}\chi_{\widetilde{\mathcal{O}}_2} \\ &\quad - 2(x^\alpha\Lambda_x z)_x + (x^\alpha\Lambda_x)_x z + a_2\widetilde{Z}\big]\mathrm{d}t + \widetilde{Z}\mathrm{d}W(t). \end{aligned} \tag{C.1}$$

Applying Lemma 3.4 to (C.1) with weight function θ_1, it holds that

$$\begin{aligned}&\mathbb{E}\int_Q \theta_1^2\left(\lambda^3\gamma^3x^{2-\alpha}\widetilde{z}^2+\lambda\gamma x^\alpha\widetilde{z}_x^2\right)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\theta_1^2\big[\lambda^3\gamma^3\chi_{\omega_1}\widetilde{z}^2+\widetilde{z}^2+\Lambda^2(q_1^2+q_2^2+\lambda^2\gamma^2x^{3\alpha}(q_{1,x}^2+q_{2,x}^2))\\&\qquad+x^{4\alpha}\Lambda_x^2(q_{1,x}^2+q_{2,x}^2)+\lambda^2\gamma^2x^\alpha\Lambda_x^2z^2+(x^\alpha\Lambda_x)_x^2z^2+\widetilde{Z}^2+\lambda^2\gamma^2\widetilde{Z}^2\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Similar to the proof of (B.2) and (B.3), by Hardy's inequality, we have that

$$\mathbb{E}\int_Q\theta_1^2\gamma\widetilde{z}^2\mathrm{d}x\mathrm{d}t\le C\mathbb{E}\int_Q\theta_1^2\gamma\big(\lambda^2\gamma^2x^{2-\alpha}\widetilde{z}^2+x^\alpha\widetilde{z}_x^2\big)\mathrm{d}x\mathrm{d}t.$$

Then, for a sufficiently large $\lambda>0$,

$$\begin{aligned}&\mathbb{E}\int_Q\theta_1^2\big(\lambda^2\gamma^3x^{2-\alpha}\widetilde{z}^2+\gamma\widetilde{z}^2\big)\mathrm{d}x\mathrm{d}t\le C\mathbb{E}\int_Q\theta_1^2\big(\lambda^2\gamma^3x^{2-\alpha}\widetilde{z}^2+\gamma x^\alpha\widetilde{z}_x^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\theta_1^2\Big[\lambda^2\gamma^3\widetilde{z}^2\chi_{\omega_1}+(q_1^2+q_2^2+\lambda\gamma^2x^{3\alpha}(q_{1,x}^2+q_{2,x}^2))\chi_{(0,1)\setminus\widetilde{G}_0}\\&\qquad+\lambda\gamma^2Z^2\chi_{(0,1)\setminus\widetilde{G}_0}+\lambda\gamma^2x^\alpha z^2\chi_{G_0\setminus\widetilde{G}_0}\Big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By the definition of Λ and $\omega_1\subseteq G_0$, one can see that

$$\begin{aligned}&\mathbb{E}\int_Q\theta_1^2(\lambda^2\gamma^3x^{2-\alpha}z^2+\gamma z^2)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\theta_1^2\big[\lambda^2\gamma^3z^2x^{2-\alpha}\chi_{G_0}+\gamma z^2\chi_{G_0}+\lambda^2\gamma^3z^2\chi_{\omega_1}+\lambda\gamma^2x^\alpha z^2\chi_{G_0\setminus\widetilde{G}_0}\\&\qquad+(q_1^2+q_2^2+\lambda\gamma^2x^{3\alpha}(q_{1,x}^2+q_{2,x}^2)+\lambda\gamma^2Z^2)\chi_{(0,1)\setminus\widetilde{G}_0}\big]\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\theta_1^2\big[\lambda^2\gamma^3z^2\chi_{G_0}+(q_1^2+q_2^2+\lambda\gamma^2x^\alpha(q_{1,x}^2+q_{2,x}^2))\chi_{(0,1)\setminus\widetilde{G}_0}+\lambda\gamma^2Z^2\big]\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{C.2}$$

Here, we also assume that (H_1) holds. If (3.11) and (3.12) hold, then for $i=1,2$, applying Lemma 3.3 to the second equation of (3.4) with weight function θ_i, one can obtain that

$$\begin{aligned}&\mathbb{E}\int_Q\theta_i^2\lambda^2\gamma^2\big(\lambda^2\gamma^2x^{2-\alpha}q_i^2+x^\alpha q_{i,x}^2\big)\mathrm{d}x\mathrm{d}t\\&\quad\le C\mathbb{E}\int_Q\theta_i^2\lambda\gamma\left(\lambda^3\gamma^3q_i^2\chi_{\omega_i}+\frac{1}{\mu_i^2}z^2\chi_{G_i}+\lambda^2\gamma^2x^{2-\alpha}q_i^2+\lambda\gamma^{\frac{5}{4}}q_i^2+x^\alpha q_{i,x}^2\right)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{C.3}$$

Choosing a sufficiently large $\lambda>0$, $\mathbb{E}\int_Q\theta_i^2(\lambda^3\gamma^3x^{2-\alpha}q_i^2+\lambda\gamma x^\alpha q_{i,x}^2)\mathrm{d}x\mathrm{d}t$ can be absorbed by the left-hand side of (C.3). In the following, we give an estimate on $\mathbb{E}\int_Q\theta_i^2\lambda^2\gamma^{\frac{9}{4}}q_i^2\mathrm{d}x\mathrm{d}t$. By Hölder's inequality, for any $\epsilon>0$,

$$\begin{aligned}\mathbb{E}\int_Q\theta_i^2\lambda^2\gamma^{\frac{9}{4}}q_i^2\mathrm{d}x\mathrm{d}t&=\mathbb{E}\int_Q(\theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\frac{\alpha-2}{3}}q_i^2)^{\frac{3}{4}}(\theta_i^2\lambda^4\gamma^4x^{2-\alpha}q_i^2)^{\frac{1}{4}}\mathrm{d}x\mathrm{d}t\\&\le\epsilon\mathbb{E}\int_Q\theta_i^2\lambda^4\gamma^4x^{2-\alpha}q_i^2\mathrm{d}x\mathrm{d}t+C\mathbb{E}\int_Q\theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\frac{\alpha-2}{3}}q_i^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{C.4}$$

Further, by Hardy's inequality, when $\alpha \neq 1$,

$$\begin{aligned}&\mathbb{E}\int_Q \theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\frac{\alpha-2}{3}}q_i^2\mathrm{d}x\mathrm{d}t \leq \mathbb{E}\int_Q \theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\alpha-2}q_i^2\mathrm{d}x\mathrm{d}t\\ &\quad\leq C\mathbb{E}\int_Q \lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\alpha}(\theta_i q_i)_x^2\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \left(\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\alpha}\theta_i^2 q_{i,x}^2 + \lambda^{\frac{10}{3}}\gamma^{\frac{11}{3}}x^{\alpha}\theta_i^2 q_i^2\right)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

and when $\alpha = 1$,

$$\begin{aligned}\mathbb{E}\int_Q \theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\frac{\alpha-2}{3}}q_i^2\mathrm{d}x\mathrm{d}t &= \mathbb{E}\int_Q \theta_i^2\lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{-\frac{1}{3}}q_i^2\mathrm{d}x\mathrm{d}t\\ &\leq C\mathbb{E}\int_Q \lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\frac{5}{3}}(\theta_i q_i)_x^2\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\alpha}(\theta_i q_i)_x^2\mathrm{d}x\mathrm{d}t\\ &\leq C\mathbb{E}\int_Q \lambda^{\frac{4}{3}}\gamma^{\frac{5}{3}}x^{\alpha}\theta_i^2 q_{i,x}^2\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_Q \lambda^{\frac{10}{3}}\gamma^{\frac{11}{3}}x^{\alpha}\theta_i^2 q_i^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This together with (C.3) and (C.4) implies that

(C.5)
$$\mathbb{E}\int_Q \theta_i^2\lambda^2\gamma^2\left(\lambda^2\gamma^2x^{2-\alpha}q_i^2 + x^{\alpha}q_{i,x}^2\right)\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \theta_i^2\lambda\gamma\left(\lambda^3\gamma^3 q_i^2\chi_{\omega_i} + \frac{z^2}{\mu_i^2}\chi_{G_i}\right)\mathrm{d}x\mathrm{d}t.$$

Summing (C.2) and (C.5), by Hardy's inequality and $\theta_1 = \theta_2$ in $((0,1)\setminus \widetilde{G}_0)\times(0,T)$, we find that for a sufficiently large $\lambda > 0$,

$$\begin{aligned}&\mathbb{E}\int_Q \big[\theta_1^2(\lambda^2\gamma^3x^{2-\alpha}z^2 + \lambda^4\gamma^4x^{2-\alpha}q_1^2 + \lambda^2\gamma^2x^{\alpha}q_{1,x}^2) + \theta_1^2\gamma z^2\\ &\qquad + \theta_2^2(\lambda^4\gamma^4x^{2-\alpha}q_2^2 + \lambda^2\gamma^2x^{\alpha}q_{2,x}^2)\big]\mathrm{d}x\mathrm{d}t\\ &\quad\leq C\mathbb{E}\int_Q \left[\theta_1^2\left(\lambda^2\gamma^3z^2\chi_{G_0} + \lambda\gamma^2Z^2 + \left(\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}\right)\lambda\gamma z^2 + \lambda^4\gamma^4q_1^2\chi_{\omega_1}\right)\right.\\ &\qquad \left.+ \theta_2^2(\lambda\gamma z^2\chi_{G_0} + \lambda^4\gamma^4q_2^2\chi_{\omega_2})\right]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This implies that for sufficiently large $\mu_1, \mu_2 > 0$ (such as $\mu_1, \mu_2 > \lambda$),

(C.6)
$$\begin{aligned}&\mathbb{E}\int_Q \big[\theta_1^2\left(\lambda^2\gamma^3x^{2-\alpha}z^2 + \lambda^4\gamma^4x^{2-\alpha}q_1^2 + \lambda^2\gamma^2x^{\alpha}q_{1,x}^2 + \gamma z^2\right)\\ &\quad + \theta_2^2\left(\lambda^4\gamma^4x^{2-\alpha}q_2^2 + \lambda^2\gamma^2x^{\alpha}q_{2,x}^2\right)\big]\mathrm{d}x\mathrm{d}t\\ &\quad\leq C\mathbb{E}\int_Q \big[\lambda^4\gamma^4\left(\theta_1^2q_1^2\chi_{\omega_1} + \theta_2^2q_2^2\chi_{\omega_2}\right) + \left(\theta_1^2\lambda^2\gamma^3 + \theta_2^2\lambda\gamma\right)z^2\chi_{G_0} + \theta_1^2\lambda\gamma^2Z^2\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

In order to estimate the terms $\mathbb{E}\int_Q \lambda^4\gamma^4(\theta_1^2q_1^2\chi_{\omega_1} + \theta_2^2q_2^2\chi_{\omega_2})\mathrm{d}x\mathrm{d}t$, for $i = 1, 2$, we choose a function $\xi_i \in C_0^\infty(\mathbb{R})$ satisfying that $0 \leq \xi_i \leq 1$, $\mathrm{supp}\xi_i \subseteq \mathcal{O}_i'$, and $\xi_i = 1$ in ω_i. Then, from $\mathcal{O}_1' \cap \mathcal{O}_2 = \emptyset$, $\mathcal{O}_1' \cap \widetilde{\mathcal{O}}_i = \emptyset$, $\mathcal{O}_2' \cap \mathcal{O}_1 = \emptyset$, and $\mathcal{O}_2' \cap \widetilde{\mathcal{O}}_i = \emptyset$, similar to (B.7), for any $\epsilon > 0$, we have

(C.7)
$$\mathbb{E}\int_Q \theta_i^2\lambda^4\gamma^4q_i^2\chi_{\omega_i}\mathrm{d}x\mathrm{d}t \leq C\mathbb{E}\int_Q \theta_i^2\left(\lambda^8\gamma^8z^2\chi_{\mathcal{O}_i'} + \epsilon\lambda^4\gamma^4x^{2-\alpha}q_i^2 + \epsilon\lambda^2\gamma^2x^{\alpha}q_{i,x}^2\right)\mathrm{d}x\mathrm{d}t.$$

From (C.6)–(C.7) and $\mathcal{O}_i' \subset G_0 (i=1,2)$, it holds that

$$\begin{aligned}\text{(C.8)}\quad &\mathbb{E}\int_Q \big[\theta_1^2(\lambda^2\gamma^3 x^{2-\alpha} z^2 + \lambda^4\gamma^4 x^{2-\alpha} q_1^2 + \lambda^2\gamma^2 x^\alpha q_{1,x}^2 + \gamma z^2)\\ &\quad + \theta_2^2(\lambda^4\gamma^4 x^{2-\alpha} q_2^2 + \lambda^2\gamma^2 x^\alpha q_{2,x}^2)\big]\mathrm{d}x\mathrm{d}t\\ &\le C\mathbb{E}\int_Q \big[\theta_1^2(\lambda^8\gamma^8 z^2\chi_{G_0} + \lambda\gamma^2 Z^2) + \theta_2^2\lambda^8\gamma^8 z^2\chi_{G_0}\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

If (3.11) and (3.13) hold, then, by applying Lemma 3.3 to the second equation of (3.4) with weight function θ_1, one can obtain that

(C.9)

$$\mathbb{E}\int_Q \theta_1^2\lambda^2\gamma^2\big(\lambda^2\gamma^2 x^{2-\alpha} q_1^2 + x^\alpha q_{1,x}^2\big)\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta_1^2\lambda\gamma\left(\lambda^3\gamma^3 q_1^2\chi_{\omega_1} + \frac{z^2}{\mu_1^2}\chi_{G_1}\right)\mathrm{d}x\mathrm{d}t.$$

Similar to (C.7), choosing a function $\xi_3 \in C_0^\infty(\mathbb{R})$ such that $0 \le \xi_3 \le 1$, $\mathrm{supp}\xi_3 \subseteq \mathcal{O}_3'$, and $\xi_3 = 1$ in ω_1, noting that $\mathcal{O}_3' \subset ((\mathcal{O}_1 \cap G_0) \setminus \mathcal{O}_2)$ and $\mathcal{O}_3' \cap \widetilde{\mathcal{O}}_i = \emptyset$, we have

(C.10)

$$\mathbb{E}\int_Q \theta_1^2\lambda^4\gamma^4 q_1^2\chi_{\omega_1}\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta_1^2\big(\lambda^8\gamma^8 z^2\chi_{G_0} + \epsilon\lambda^4\gamma^4 x^{2-\alpha} q_1^2 + \epsilon\lambda^2\gamma^2 x^\alpha q_{1,x}^2\big)\mathrm{d}x\mathrm{d}t.$$

From (C.9) and (C.10), one can get that

$$\mathbb{E}\int_Q \theta_1^2\lambda^2\gamma^2\big(\lambda^2\gamma^2 x^{2-\alpha} q_1^2 + x^\alpha q_{1,x}^2\big)\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta_1^2\left(\lambda^8\gamma^8\chi_{G_0} + \frac{\lambda\gamma}{\mu_1^2}\chi_{G_1}\right)z^2\mathrm{d}x\mathrm{d}t.$$

This together with (C.2) indicates that for a sufficiently large $\mu_1 > 0$,

$$\begin{aligned}\text{(C.11)}\quad &\mathbb{E}\int_Q \theta_1^2(\lambda^2\gamma^3 x^{2-\alpha} z^2 + \gamma z^2 + \lambda^4\gamma^4 x^{2-\alpha} q_1^2 + \lambda^2\gamma^2 x^\alpha q_{1,x}^2)\mathrm{d}x\mathrm{d}t\\ &\le C\mathbb{E}\int_Q \theta_1^2\big[\lambda^8\gamma^8 z^2\chi_{G_0} + \lambda\gamma^2 Z^2 + (q_2^2 + \lambda\gamma^2 x^{3\alpha} q_{2,x}^2)\chi_{(0,1)\setminus\widetilde{G}_0}\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

To estimate the last term of (C.11), we borrow some ideas from the proof of Proposition 3.5. Let $\eta = \alpha_1 q_1 + \alpha_2 q_2$. Then, one can obtain an estimate similar to (B.5),

$$\begin{aligned}&\mathbb{E}\int_Q \theta_2^2\lambda^2\gamma^2\big(\lambda^2\gamma^2 x^{2-\alpha}\eta^2 + x^\alpha\eta_x^2\big)\mathrm{d}x\mathrm{d}t\\ &\quad\le C\mathbb{E}\int_Q \theta_2^2\lambda\gamma\Big[\lambda^3\gamma^3\eta^2\chi_{\omega_2} + \Big(\frac{\alpha_1}{\mu_1}\chi_{G_1} + \frac{\alpha_2}{\mu_2}\chi_{G_2}\Big)^2 z^2\Big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Similar to (C.7) and (C.10), choosing a function $\xi_4 \in C_0^\infty(\mathbb{R})$ such that $0 \le \xi_4 \le 1$, $\mathrm{supp}\xi_4 \subseteq \mathcal{O}_4'$, and $\xi_4 = 1$ in ω_2, noting that $\mathcal{O}_4' \subset (\mathcal{O}_1 \cap \mathcal{O}_2 \cap G_0)$ and $\mathcal{O}_4' \cap \widetilde{\mathcal{O}}_i = \emptyset$, we have

$$\mathbb{E}\int_Q \theta_2^2\lambda^4\gamma^4\eta^2\chi_{\omega_2}\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta_2^2\big(\lambda^8\gamma^8 z^2\chi_{G_0} + \epsilon\lambda^4\gamma^4 x^{2-\alpha}\eta^2 + \epsilon\lambda^2\gamma^2 x^\alpha\eta_x^2\big)\mathrm{d}x\mathrm{d}t.$$

Then,

$$\mathbb{E}\int_Q \theta_2^2\lambda^2\gamma^2\big(\lambda^2\gamma^2 x^{2-\alpha}\eta^2 + x^\alpha\eta_x^2\big)\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_Q \theta_2^2\left[\lambda^8\gamma^8 z^2\chi_{G_0} + \lambda\gamma\left(\frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2}\right)^2 z^2\right]\mathrm{d}x\mathrm{d}t.$$

Combining the above inequality with (C.11), from $\theta_1=\theta_2$ in $((0,1)\setminus\widetilde{G}_0)\times(0,T)$ and $|b|^2\le 2(|a|^2+|a+b|^2)$, it follows that for sufficiently large $\mu_1,\mu_2>0$,

$$\begin{aligned}&\mathbb{E}\int_Q\big[\theta_1^2\lambda^2\gamma^3x^{2-\alpha}z^2+\theta_1^2\gamma z^2+\lambda^4\gamma^4x^{2-\alpha}(\theta_1^2q_1^2+\theta_2^2\eta^2+\theta_1^2q_2^2\chi_{(0,1)\setminus\widetilde{G}_0})\\&\quad+\lambda^2\gamma^2x^\alpha(\theta_1^2q_{1,x}^2+\theta_2^2\eta_x^2+\theta_1^2q_{2,x}^2\chi_{(0,1)\setminus\widetilde{G}_0})\big]\mathrm{d}x\mathrm{d}t\\&\le C\mathbb{E}\int_Q\big[\theta_1^2\lambda^2\gamma^3x^{2-\alpha}z^2+\theta_1^2\gamma z^2+\lambda^4\gamma^4x^{2-\alpha}(\theta_1^2q_1^2+\theta_2^2\eta^2)+\lambda^2\gamma^2x^\alpha(\theta_1^2q_{1,x}^2+\theta_2^2\eta_x^2)\big]\mathrm{d}x\mathrm{d}t\\&\le C\mathbb{E}\int_Q\big[(\theta_1^2+\theta_2^2)\lambda^8\gamma^8z^2\chi_{G_0}+\theta_1^2\lambda\gamma^2Z^2+\theta_1^2(q_2^2+\lambda\gamma^2x^{3\alpha}q_{2,x}^2)\chi_{(0,1)\setminus\widetilde{G}_0}\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This implies that

$$\begin{aligned}&\mathbb{E}\int_Q\hat\theta^2\big[\lambda^2\gamma^3x^{2-\alpha}z^2+\gamma z^2+\lambda^4\gamma^4x^{2-\alpha}(q_1^2+q_2^2)+\lambda^2\gamma^2x^\alpha(q_{1,x}^2+q_{2,x}^2)\big]\mathrm{d}x\mathrm{d}t\\&\le C\mathbb{E}\int_Q\hat\theta^2\big[\lambda^2\gamma^3x^{2-\alpha}z^2+\gamma z^2+\lambda^4\gamma^4x^{2-\alpha}(q_1^2+\eta^2)+\lambda^2\gamma^2x^\alpha(q_{1,x}^2+\eta_x^2)\big]\mathrm{d}x\mathrm{d}t\\&\le C\mathbb{E}\int_Q\big[(\theta_1^2+\theta_2^2)\lambda^8\gamma^8z^2\chi_{G_0}+\theta_1^2\lambda\gamma^2Z^2\big]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

This and (C.8) give the desired result. ∎

Appendix D. Proof of Theorem 4.1. First, a weighted identity for the stochastic degenerate parabolic operator $\mathrm{d}y-(x^\alpha y_x)_x\mathrm{d}t$ is given. Define an unbounded operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subseteq L^2(0,1)\to L^2(0,1)$ as follows:

$$\mathcal{A}y=(x^\alpha y_x)_x,\quad \mathcal{D}(\mathcal{A})=\{y\in H^1_\alpha(0,1)|x^\alpha y_x\in L^2(0,1)\}.$$

LEMMA D. *Let y be a $\mathcal{D}(\mathcal{A})$-valued continuous semimartingale, and set $v=\widetilde\theta y$. Then, for a.e. $(x,t)\in Q$, and $\mathbb{P}$-a.s. $\omega\in\Omega$, one has the following weighted identity:*

$$\begin{aligned}&\widetilde\theta\left[-(x^\alpha v_x)_x-s\lambda\varphi\widetilde\psi_tv+\frac14\lambda v\right]\left[\mathrm{d}y-(x^\alpha y_x)_x\mathrm{d}t\right]\\&\quad=\big[(x^\alpha v_x)_x+s\lambda\varphi\widetilde\psi_tv\big]^2\mathrm{d}t+\frac12\mathrm{d}\Big(-s\lambda\varphi\widetilde\psi_tv^2+x^\alpha v_x^2+\frac14\lambda v^2\Big)\\&\qquad-(x^\alpha v_x\mathrm{d}v)_x-\frac12x^\alpha(\mathrm{d}v_x)^2-\frac14(\lambda x^\alpha v_xv)_x\mathrm{d}t+\frac12s\lambda\varphi\widetilde\psi_t(\mathrm{d}v)^2\\&\qquad-\frac18\lambda(\mathrm{d}v)^2+\frac12\Big[(s\lambda\varphi\widetilde\psi_t)_t-\frac12s\lambda^2\varphi\widetilde\psi_t\Big]v^2\mathrm{d}t+\frac14\lambda x^\alpha v_x^2\mathrm{d}t.\end{aligned}\tag{D.1}$$

Proof. By Itô's formula, we have

$$\begin{aligned}&\widetilde\theta\big[-(x^\alpha v_x)_x-s\lambda\varphi\psi_tv\big]\big[\mathrm{d}y-(x^\alpha y_x)_x\mathrm{d}t\big]\\&\quad=\big[-(x^\alpha v_x)_x-s\lambda\varphi\widetilde\psi_tv\big]\big[\mathrm{d}v-s\lambda\varphi\widetilde\psi_tv\mathrm{d}t-(x^\alpha v_x)_x\mathrm{d}t\big]\\&\quad=-(x^\alpha v_x\mathrm{d}v)_x+\frac12x^\alpha\mathrm{d}(v_x^2)-\frac12x^\alpha(\mathrm{d}v_x)^2-\frac12\mathrm{d}(s\lambda\varphi\widetilde\psi_tv^2)\\&\qquad+\frac12(s\lambda\varphi\widetilde\psi_t)_tv^2\mathrm{d}t+\frac12s\lambda\varphi\widetilde\psi_t(\mathrm{d}v)^2+\big[(x^\alpha v_x)_x+s\lambda\varphi\widetilde\psi_tv\big]^2\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\frac14\lambda\widetilde\theta v\,[\mathrm{d}y-(x^\alpha y_x)_x\mathrm{d}t]=\frac14\lambda v\big[\mathrm{d}v-s\lambda\varphi\widetilde\psi_tv\mathrm{d}t-(x^\alpha v_x)_x\mathrm{d}t\big]\\&\quad=\frac18\mathrm{d}(\lambda v^2)-\frac18\lambda(\mathrm{d}v)^2-\frac14s\lambda^2\varphi\widetilde\psi_tv^2\mathrm{d}t-\frac14(\lambda x^\alpha v_xv)_x\mathrm{d}t+\frac14\lambda x^\alpha v_x^2\mathrm{d}t.\end{aligned}$$

Combining the above two equalities leads to the desired equality (D.1). ∎

Now, we prove the Carleman estimate (4.2).

Proof of Theorem 4.1. Applying Lemma D to (3.6), integrating the equality (D.1) on Q, and taking mathematical expectation, we can get

$$\begin{aligned}&4\mathbb{E}\int_Q \widetilde{\theta}\left[-(x^\alpha v_x)_x - s\lambda\varphi\widetilde{\psi}_t v + \frac{1}{4}\lambda v\right]\left[\mathrm{d}y - (x^\alpha y_x)_x \mathrm{d}t\right]\mathrm{d}x\\ &= 4\mathbb{E}\int_Q \left[(x^\alpha v_x)_x + s\lambda\varphi\widetilde{\psi}_t v\right]^2 \mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q \left[s\lambda\varphi\widetilde{\psi}_t(\mathrm{d}v)^2 - \lambda(\mathrm{d}v)^2 - x^\alpha(\mathrm{d}v_x)^2\right]\mathrm{d}x\\ &\quad + 2\mathbb{E}\int_Q \mathrm{d}\left(x^\alpha v_x^2 - s\lambda\varphi\widetilde{\psi}_t v^2 + \frac{\lambda v^2}{4}\right)\mathrm{d}x + \mathbb{E}\int_Q \left[2(s\lambda\varphi\widetilde{\psi}_t)_t v^2 - s\lambda^2\varphi\widetilde{\psi}_t v^2 + \lambda x^\alpha v_x^2\right]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now, we evaluate the right-hand side of the above equality. For the second term, by (3.6) and Itô's formula, it follows that

$$\begin{aligned}&\mathbb{E}\int_Q \left[s\lambda\varphi\widetilde{\psi}_t(\mathrm{d}v)^2 - \frac{1}{4}\lambda(\mathrm{d}v)^2 - x^\alpha(\mathrm{d}v_x)^2\right]\mathrm{d}x\\ &\quad = \mathbb{E}\int_Q \left[s\lambda\varphi\widetilde{\theta}^2 g^2 - \frac{1}{4}\lambda\widetilde{\theta}^2 g^2 - x^\alpha(\widetilde{\theta}g_x)^2\right]\mathrm{d}x\mathrm{d}t \geq C\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda g^2 - x^\alpha g_x^2\big)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{D.2}$$

For the third term, recalling that $\widetilde{\psi}_t = 1$, it holds that

$$\mathbb{E}\int_Q \mathrm{d}\left(x^\alpha v_x^2 - s\lambda\varphi\widetilde{\psi}_t v^2 + \frac{1}{4}\lambda v^2\right)\mathrm{d}x \geq -C\mathbb{E}\int_0^1 \big(s\lambda v^2(T) + x^\alpha v_x^2(0)\big)\mathrm{d}x. \tag{D.3}$$

For the fourth term, we have

$$\mathbb{E}\int_Q \left[2(s\lambda\varphi\widetilde{\psi}_t)_t v^2 - s\lambda^2\varphi\widetilde{\psi}_t v^2 + \lambda x^\alpha v_x^2\right]\mathrm{d}x\mathrm{d}t \geq C\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 y^2 + \lambda x^\alpha y_x^2\big)\mathrm{d}x\mathrm{d}t.$$

This together with (D.2)–(D.3) gives the desired inequality. ∎

Appendix E. Proof of Theorem 4.3. Based on (4.3), we prove Theorem 4.3 by a duality method. To achieve this goal, introduce the following forward stochastic degenerate parabolic equation:

$$\begin{cases} \mathrm{d}y - \big(x^\alpha y_x\big)_x \mathrm{d}t = \big[\widetilde{\theta}^2 s\lambda^2 h - (\widetilde{\theta}^2\lambda x^\alpha h_x)_x + \chi_{G_3} u_3\big]\mathrm{d}t & \\ \qquad\qquad\qquad + (v_3 + \widetilde{\theta}^2 s\lambda H)\mathrm{d}W(t) & \text{in } Q,\\ \begin{cases} y(0,t) = 0 & \text{if } 0 \leq \alpha < 1,\\ (x^\alpha y_x)(0,t) = 0 & \text{if } 1 \leq \alpha < 2\end{cases} & \text{on } (0,T),\\ y(1,t) = 0 & \text{on } (0,T),\\ y(x,0) = 0 & \text{in } (0,1),\end{cases} \tag{E.1}$$

where (h, H) is any given solution to (3.8), (u_3, v_3) is the control variable, y is the state variable, and G_3 is an arbitrary subdomain of $(0,1)$. Then, we have the following controllability and inequality for (E.1).

PROPOSITION E. *There exist a pair of control* $(\widehat{u}_3, \widehat{v}_3) \in U_3 \triangleq L^2_{\mathbb{F}}(0,T;L^2(G_3)) \times L^2_{\mathbb{F}}(0,T;L^2(0,1))$ *such that* (E.1) *admits a solution* $\widehat{y} \in \mathcal{H}$ *satisfying* $\widehat{y}(x,T) = 0$ *in* $(0,1)$, $\mathbb{P}$*-a.s. Moreover,*

$$\begin{aligned}&\mathbb{E}\int_Q \widetilde{\theta}^{-2}\big(\widehat{y}^2 + s^{-1}\lambda^{-1}x^{-\alpha}\widehat{y}_x^2 + \chi_{G_3}\lambda^{-1}\widehat{u}_3^2 + \widehat{v}_3^2\big)\mathrm{d}x\mathrm{d}t\\ &\quad \leq C\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2 h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{E.2}$$

Proof. For any $\varepsilon > 0$, construct the following optimal control problem $(\mathbf{P}_\varepsilon)$:

$$\min_{(u_3(\cdot),v_3(\cdot))\in\mathcal{U}} \left\{ \frac{1}{2}\mathbb{E}\int_Q \widetilde{\theta}^{-2}\big(y^2 + \lambda^{-1}\chi_{G_3}u_3^2 + v_3^2\big)\mathrm{d}x\mathrm{d}t + \frac{1}{2\varepsilon}\mathbb{E}\int_0^1 y^2(T)\mathrm{d}x \right\},$$

subject to (E.1), where $\mathcal{U} = \big\{(u_3, v_3) \in U_3 \big|\ \mathbb{E}\int_Q \widetilde{\theta}^{-2}(\chi_{G_3}u_3^2 + v_3^2)\mathrm{d}x\mathrm{d}t < \infty\ \big\}$. Then, one can check that for any $\varepsilon > 0$, the above optimal control problem $(\mathbf{P}_\varepsilon)$ admits a unique optimal solution $(u_{3,\varepsilon}, v_{3,\varepsilon}, y_\varepsilon) \in \mathcal{U} \times L^2_{\mathbb{F}}(0,T;H^1_\alpha(0,1))$. Also, $u_{3,\varepsilon} = \widetilde{\theta}^2\lambda z_\varepsilon\chi_{G_3}$ and $v_{3,\varepsilon} = \widetilde{\theta}^2 Z_\varepsilon$ in Q, $\mathbb{P}$-*a.s.*, where $(z_\varepsilon, Z_\varepsilon)$ satisfies the following equation:

$$\text{(E.3)}\qquad \begin{cases} \mathrm{d}z_\varepsilon + \big(x^\alpha z_{\varepsilon,x}\big)_x\mathrm{d}t = \widetilde{\theta}^{-2}y_\varepsilon\mathrm{d}t + Z_\varepsilon\mathrm{d}W(t) & \text{in } Q,\\ \begin{cases} z_\varepsilon(0,t) = 0 & \text{if } 0 \le \alpha < 1,\\ \big(x^\alpha z_{\varepsilon,x}\big)(0,t) = 0 & \text{if } 1 \le \alpha < 2 \end{cases} & \text{on } (0,T),\\ z_\varepsilon(1,t) = 0 & \text{on } (0,T),\\ z_\varepsilon(x,0) = -\frac{1}{\varepsilon}y_\varepsilon(x,T) & \text{in } (0,1). \end{cases}$$

Next, a uniform estimate for the optimal solutions $\{(u_{3,\varepsilon}, v_{3,\varepsilon}, y_\varepsilon)\}_{\varepsilon>0}$ will be established. By (E.1), (E.3), and Itô's formula, it follows that

$$\begin{aligned} &\mathbb{E}\int_0^1 y_\varepsilon(T)z_\varepsilon(T)\mathrm{d}x = \mathbb{E}\int_Q \mathrm{d}(y_\varepsilon z_\varepsilon)\mathrm{d}x\\ &\quad= \mathbb{E}\int_Q \Big(\widetilde{\theta}^{-2}y_\varepsilon^2 + \widetilde{\theta}^2\lambda x^\alpha h_x z_{\varepsilon,x} + \widetilde{\theta}^2 s\lambda^2 h z_\varepsilon + \chi_{G_3}\widetilde{\theta}^2\lambda z_\varepsilon^2 + \widetilde{\theta}^2 Z_\varepsilon^2 + \widetilde{\theta}^2 s\lambda H Z_\varepsilon\Big)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

This together with (4.3) indicates that for a sufficiently small $\epsilon > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q \big(\widetilde{\theta}^{-2}y_\varepsilon^2 + \chi_{G_3}\widetilde{\theta}^2\lambda z_\varepsilon^2 + \widetilde{\theta}^2 Z_\varepsilon^2\big)\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_0^1 y_\varepsilon^2(T)\mathrm{d}x\\ &\quad\le \epsilon\mathbb{E}\int_Q \widetilde{\theta}^2 s\lambda^2 z_\varepsilon^2\mathrm{d}x\mathrm{d}t + C(\epsilon)\mathbb{E}\int_Q \widetilde{\theta}^2 s\lambda^2 h^2\mathrm{d}x\mathrm{d}t + \epsilon\mathbb{E}\int_Q \widetilde{\theta}^2\lambda x^\alpha z_{\varepsilon,x}^2\mathrm{d}x\mathrm{d}t\\ &\qquad + C(\epsilon)\mathbb{E}\int_Q \widetilde{\theta}^2\lambda x^\alpha h_x^2\mathrm{d}x\mathrm{d}t + \epsilon\mathbb{E}\int_Q \widetilde{\theta}^2 s\lambda Z_\varepsilon^2\mathrm{d}x\mathrm{d}t + C(\epsilon)\mathbb{E}\int_Q \widetilde{\theta}^2 s\lambda H^2\mathrm{d}x\mathrm{d}t\\ &\quad\le C\epsilon\mathbb{E}\int_Q \widetilde{\theta}^2\big(\widetilde{\theta}^{-2}y_\varepsilon\big)^2\mathrm{d}x\mathrm{d}t + C(\epsilon)\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

which implies that

$$\text{(E.4)}\qquad \begin{aligned} &\mathbb{E}\int_Q \widetilde{\theta}^{-2}\big(y_\varepsilon^2 + \chi_{G_3}\lambda^{-1}u_{3,\varepsilon}^2 + v_{3,\varepsilon}^2\big)\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_0^1 y_\varepsilon^2(T)\mathrm{d}x\\ &\quad\le C\mathbb{E}\int_Q \widetilde{\theta}^2\big(s\lambda^2h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Moreover, by the first equation of (E.1) and Itô's formula, we find that for any $\epsilon > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q \widetilde{\theta}^{-2}s^{-1}\lambda^{-1}x^\alpha y_{\varepsilon,x}^2\mathrm{d}x\mathrm{d}t\\ &\quad\le C\bigg[\mathbb{E}\int_Q \widetilde{\theta}^{-2}y_\varepsilon^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q s^{-1}x^\alpha|h_x y_{\varepsilon,x}|\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \lambda|y_\varepsilon h|\mathrm{d}x\mathrm{d}t\\ &\qquad + \mathbb{E}\int_Q \widetilde{\theta}^{-2}s^{-1}\lambda^{-1}\chi_{G_3}|y_\varepsilon u_{3,\varepsilon}|\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \big(\widetilde{\theta}^{-2}s^{-1}\lambda^{-1}v_{3,\varepsilon}^2 + \widetilde{\theta}^2 s\lambda H^2\big)\mathrm{d}x\mathrm{d}t\bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon\mathbb{E}\int_Q \big(\widetilde{\theta}^{-2}s^{-2}\lambda^{-1}x^\alpha y_{\varepsilon,x}^2 + \widetilde{\theta}^{-2}y_\varepsilon^2 + \widetilde{\theta}^{-2}s^{-2}\lambda^{-1}\chi_{G_3}y_\varepsilon^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad + C(\epsilon)\mathbb{E}\int_Q \big(\widetilde{\theta}^{2}\lambda x^\alpha h_x^2 + \widetilde{\theta}^{2}\lambda^2 h^2 + \widetilde{\theta}^{-2}\lambda^{-1}\chi_{G_3}u_{3,\varepsilon}^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad + C\mathbb{E}\int_Q \big(\widetilde{\theta}^{-2}y_\varepsilon^2 + \widetilde{\theta}^{-2}s^{-1}\lambda^{-1}v_{3,\varepsilon}^2 + \widetilde{\theta}^{2}s\lambda H^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This together with (E.4) implies

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^{-2}\big(y_\varepsilon^2 + s^{-1}\lambda^{-1}x^\alpha y_{\varepsilon,x}^2 + \chi_{G_3}\lambda^{-1}u_{3,\varepsilon}^2 + v_{3,\varepsilon}^2\big)\mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon}\mathbb{E}\int_0^1 y_\varepsilon^2(T)\mathrm{d}x\\
&\leq C\mathbb{E}\int_Q \widetilde{\theta}^{2}\big(s\lambda^2 h^2 + \lambda x^\alpha h_x^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{E.5}
$$

Therefore, there exist a subsequence of $\{(u_{3,\varepsilon}, v_{3,\varepsilon}, y_\varepsilon)\}_{\varepsilon>0}$ (still denoted by itself) and $(\widehat{u}_3, \widehat{v}_3, \widehat{y}) \in U_3 \times L^2_{\mathbb{F}}(0,T;L^2(0,1))$ such that as $\varepsilon \to 0$,

$$
\begin{aligned}
u_{3,\varepsilon} &\to \widehat{u}_3 \qquad \text{weakly in } L^2((0,T)\times\Omega;L^2(G_3));\\
v_{3,\varepsilon} &\to \widehat{v}_3 \qquad \text{weakly in } L^2((0,T)\times\Omega;L^2(0,1));\\
y_\varepsilon &\to \widehat{y} \qquad \text{weakly in } L^2((0,T)\times\Omega;L^2(0,1)).
\end{aligned}
$$

It is easy to show that $\widehat{y}$ is a solution of (E.1) associated to $(u_3,v_3) = (\widehat{u}_3,\widehat{v}_3)$. Also, by (E.5), we get that $\widehat{y}(T) = 0$ in $(0,1)$, $\mathbb{P}$-*a.s.*, and (E.2) holds. □

Now, we prove the Carleman estimate (4.4) based on the inequality (E.2).

Proof of Theorem 4.3. For any $f_1, a^{-\frac{1}{2}}f_2 \in L^2_{\mathbb{F}}(0,T;L^2(0,1))$, let (h,H) be the corresponding solution to (3.8). Denote by $\widehat{y}$ the solution of (E.1) associated to $(\widehat{u}_3,\widehat{v}_3)$, which are the control mentioned in Proposition E. Then, by Itô's formula,

$$
\mathbb{E}\int_Q \widetilde{\theta}^{2}\big(\lambda x^\alpha h_x^2 + s\lambda^2 h^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t = -\mathbb{E}\int_Q \big(\chi_{G_3}\widehat{u}_3 h + \widehat{y}f_1 + \widehat{v}_3 H - \widehat{y}_x f_2\big)\mathrm{d}x\mathrm{d}t.
$$

It follows that for a sufficiently small $\epsilon > 0$ and a sufficiently large $\lambda > 0$,

$$
\begin{aligned}
&\mathbb{E}\int_Q \widetilde{\theta}^{2}\big(\lambda x^\alpha h_x^2 + s\lambda^2 h^2 + s\lambda H^2\big)\mathrm{d}x\mathrm{d}t\\
&\leq \epsilon\mathbb{E}\int_Q \widetilde{\theta}^{-2}\big(\chi_{G_3}\lambda^{-1}\widehat{u}_3^2 + \widehat{y}^2 + \widehat{v}_3^2 + s^{-1}\lambda^{-1}x^{-\alpha}\widehat{y}_x^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad + C\mathbb{E}\int_0^T\int_{G_3} \widetilde{\theta}^{2}\lambda h^2\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_Q \widetilde{\theta}^{2}\big(f_1^2 + H^2 + s\lambda x^\alpha f_2^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This together with (E.2) implies the desired Carleman estimate. □

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